Univrsity of California
Santa Barbara

Advances in Nonlinear Phased Arrays

A Dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy
in Electrical and Computer Engineering

by

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ABSTRACT

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For centuries the synchronization phenomena have been drawing attention in several fields of science. In the past few decades the radar industry and communication market recent motivated recent progress in microwave nonlinear phased arrays based on nonlinear synchronization phenomena. Two main types of arrays will be considered: coupled oscillator arrays (COAs) and coupled phase-locked loop arrays (CPPLAs). The first to be studied were COAs, but their limitations, mainly intrinsic small locking bandwidth, amplitude fluctuations and limited agreement between unit cells and models, drove recent efforts towards CPPLAs, which with appropriate models are more predictable than COAs and offers larger locking range and amplitude-independent phase relationships. After a brief introduction, the dissertation will describe these two types of arrays together with the recent findings associated with the improved understanding of their performances. The two offer similar advantages, such as phase-shifterless beam scanning and noise reduction, as well as analogous challenges, for example the modeling and consequent
design of unit cell and coupling schemes at microwave frequencies. In COAs advanced studies of the ideal models were able to predict the observed chaotic patterns and the transient phase evolution. Recently, in PLLAs the addition of a loop time delay enhanced the model of the single PLL far enough to predict the lower and upper boundaries for the loop gain. Additionally, the length of the coupling line together with the sign of the IF loop gain was proved to be an important factor in the transient and the steady-state phase distribution along the array. From the understanding point of view, a continuum modeling has been developed for both systems which confirms the results of the discrete analysis and highlights new features of the transient behavior of such arrays as the synchronization diffuses following a heat transfer behavior. Being governed by strongly nonlinear dynamics, still a lot needs to be understood about these synchronized arrays: the aim of these research was to show that, together with some limitations, they also present interesting properties that future research may exploit.
TABLE OF CONTENTS

Chapter 1: Introduction........................................................................................................... 1
Motivation for Coupled Oscillators Arrays (COAs)............................................................... 2
Coupled Phase-Locked Loops Arrays (CPLLAs)................................................................. 8
Synopsis of the Dissertation............................................................................................... 10

Chapter 2: Coupled Oscillator Arrays Theory: Single Element Dynamics ...................... 13
Injection Locking .................................................................................................................. 13
Dynamical Model ................................................................................................................ 18
Free-Running Oscillator .................................................................................................... 23
Injection-Locked Oscillator .............................................................................................. 30
Modulation .......................................................................................................................... 39
Amplitude Modulation ...................................................................................................... 41
Frequency Modulation ..................................................................................................... 44
Phase Modulation ............................................................................................................. 46
Time Delay Considerations .............................................................................................. 48
Noise .................................................................................................................................. 51
Free-Running..................................................................................................................... 57
Injection-Locked ............................................................................................................... 60

Chapter 3: Coupled Oscillator Arrays Theory Discrete and Continuum Modeling ............ 64
Discrete Modeling of Coupled Oscillators Arrays.............................................................. 64
Coupling Network Theory ............................................................................................... 65
Single Cell Modeling ........................................................................................................ 72
Broadband Nearest-Neighbor Coupling ........................................................................... 76
Continuum Modeling of Coupled Oscillators Arrays......................................................... 81
Derivation of Governing Equation ..................................................................................... 84
Synchronization and Steady State Solution ....................................................................... 89
A First Glimpse to the Dynamics ...................................................................................... 91

Chapter 4: Coupled Oscillator Arrays Theory: Whole Array Dynamics ......................... 99
Phase Dynamics ................................................................................................................. 99
Free-Running Arrays ....................................................................................................... 99
Scanning by Edge Detuning ............................................................................................. 110
Modulation ....................................................................................................................... 117
Modulation Schemes ....................................................................................................... 118
Stephan’s Beam Scanning ............................................................................................... 127
Time Delay Considerations ............................................................................................. 132
Phase Noise ...................................................................................................................... 135
General Case Analysis ................................................................................................. 136
Free-Running Array ........................................................................................................ 141
Globally Injected Array ................................................................................................. 143
Array with One Element Externally Locked ..................................................................... 145
Comments ......................................................................................................................... 148

Chapter 5: Coupled Oscillator Arrays: Design and Characterization ............................... 151
The Single Oscillator ....................................................................................................... 152
Design & Realization ....................................................................................................... 152
Characterization .............................................................................................................. 158
Coupling Network ......................................................................................................... 163
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>Coupled Phase-Locked Loop Arrays</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Single Cell Dynamics</td>
<td>136</td>
</tr>
<tr>
<td></td>
<td>Dynamic Equation</td>
<td>136</td>
</tr>
<tr>
<td></td>
<td>Array Dynamics</td>
<td>141</td>
</tr>
<tr>
<td></td>
<td>Single Cell Design and Characterization</td>
<td>150</td>
</tr>
<tr>
<td></td>
<td>Array Design and Characterization</td>
<td>158</td>
</tr>
<tr>
<td></td>
<td>Comments</td>
<td>166</td>
</tr>
<tr>
<td>7</td>
<td>Conclusion and Future Work</td>
<td>229</td>
</tr>
<tr>
<td>A</td>
<td>Kurokawa’s Substitution</td>
<td>232</td>
</tr>
<tr>
<td>B</td>
<td>Miscellaneous Mathematical Proofs</td>
<td>236</td>
</tr>
<tr>
<td></td>
<td>B1: Some FFT Basics for Spectral Power Analysis</td>
<td>236</td>
</tr>
<tr>
<td></td>
<td>B2: Noise of VDPO Injection-Locked to a Noiseless Source</td>
<td>241</td>
</tr>
<tr>
<td></td>
<td>B3: Noise of VDPO Injection-Locked to a Low-Noise Source</td>
<td>243</td>
</tr>
</tbody>
</table>
CHAPTER 1

Introduction

In the last two decades, the increasing need of high power systems at microwave and millimeter frequencies for the communication market motivated the research in the fields of power combining techniques. The investigation is expanding mainly in two directions.

From one side, new ways to combine amplifiers to overcome the power limitations of single devices are continuously proposed and developed. Several interesting properties of power combiners have been found, such as phase noise reduction, graceful degradation, and overall cost reduction by the possible use of smaller devices with better power added efficiency. Moreover the system loss can be significantly reduced when the summation of power occurs coherently in free space as recently shown with space power combiners.

On the opposite end, the efficiency associated with free space power combining motivated the research focused on the ability to combine efficiently several sources, such as voltage controlled oscillators or phase-locked loop synthesizers. This path didn’t lead to results useful in the power-combining field, mainly because it is
It is much easier to generate and amplify a low power modulated signal than synchronize a large number of medium power modulated sources.

However, these studies revitalized the interest in nonlinear phenomena that drive oscillating systems to lock their frequencies. A better understanding of the synchronization process, facilitated by improved mathematical models and software tools, was achieved. Consequently new useful features of these systems were found, from the phase shifterless beam scanning to the phase noise reduction. In addition a full set of nonlinear behaviors can be obtained with simple circuitry, from quasi-periodic to chaotic, from synchronized to mode-locked, offering an interesting experimental bench for the development of future communication techniques. Part of this scenario are microwave oscillator arrays, where injection locking and/or phased-locked loop techniques are used to achieve synchronous operation.

Our work focused mainly on the development of two types of microwave radiating arrays, Coupled Oscillators Arrays (COAs) and Coupled Phase-Locked Loops Arrays (CPLLAs).

**Motivation for Coupled Oscillators Arrays (COAs)**

As observed also in biological, chemical, social and other systems, the dynamical behavior of real signal sources can be affected by nearby sources close enough to interact. The strength of this interaction – defined as partial transfer of
the signal between sources – determines how much the mode of operation is influenced. A very strong interaction may lead the signals to lock together into a common state, while a very weak one will leave the sources in an unperturbed (free-running) condition. In between these two limits several other behaviors may occur based on the equations ruling the system dynamics.

The interaction is expressed mathematically through the use of coupling terms that cross-link the dynamic equations of the single sources. When the characteristic equation of the uncoupled source is nonlinear, the dynamics governing the overall system is very complex: even the mathematical tools available today are able to completely describe such systems with close solutions in simple cases. However, the advent of fast computing allowed for numerical solutions that highlight the important features of the system. In addition, a more efficient modeling of the coupling as well as the single signal source is leading to a better understanding of the phenomena that drive coupled nonlinear systems to have so many different behaviors.

From an engineering point of view, the fascinating aspect of these systems is that a better understanding of their modus operandi can be exploited in electronics, chemistry, mechanics and other scientific fields. Linear models have been well exploited in the last two centuries, but their limitations in describing real systems (such as saturation, mixing and memory) are known. The most advanced and versatile systems in the Nature work indeed as nonlinear subsystems coupled
together (think about the human body), thus the interest in implementing and understanding them is evident.

In this scenario, electrical oscillators showed promising nonlinear behaviors that can be exploited in the communication, consumer and radar industries. When these oscillators are coupled together the complexity of the whole system increases but the dynamic properties become very attractive. The interaction between the elements may occur in several ways, all of them involving the exchange of information of the state of the transmitting element, expressed as power, frequency, phase, amplitude or other. The phenomenon governing the dynamics of coupled oscillators is the injection of power. When some power is injected somehow into the circuitry of a nonlinear oscillator, its instantaneous frequency changes. Several state patterns are possible based on circuit parameters and strength of injection. When the injection signal is strong enough, the frequency locks to the injection signal.

This interesting property – called injection locking – is particularly visible when microwave sources are used, due to the fact that the power can be radiatively transferred between elements. This transfer has traditionally being considered a parasitic effect of microwave circuits, but yet stimulated the investigation of nonlinear phenomena responsible of the synchronization process in view of using them for power combining in the search of the ideal high frequency power source previously mentioned.
The study of intrinsic nonlinearities of high frequency oscillators started in the early twentieth century. One of the most influential figures was Van der Pol\textsuperscript{vii}, who was able to accurately model the oscillation phenomena using a second order nonlinear differential equation, afterward referred by mathematicians and engineers as the VDP equation. Both the free running and the injection locked (called ‘forced’ at those time) cases were analyzed. From one side his study helped the radio engineers to understand the weird behaviors observed in circuits with strongly nonlinear triodes. From the other side the VDP model captures some of the essential features of real oscillating systems, such as the amplitude dependent negative resistance. For this reason Adler\textsuperscript{viii} carried on the Van der Pol investigation of forced oscillation to derive a differential equation that relates the oscillator phase to the injection signal parameters.

The monumental work of Van der Pol was then applied and refined by Kurokawa\textsuperscript{ix} to the injection of microwave oscillators, which present a more complicated negative resistance that was pictured in VDP and Adler’s equations.

When several nonlinear oscillators are coupled together, then a set of coupled nonlinear differential equations must be found. York et al\textsuperscript{x} extended the Kurokawa’s analysis to coupled oscillators systems deriving a set of coupled nonlinear differential equations for phase and amplitude dynamics. The important of such work was enormous: till then in the microwave engineer community a lot of coupling schemes have been proposed but none of them combined reliability and
simplicity. From one side radiatively coupling is easy to implement and offers broadband coupling, but it’s extremely sensitive to the environment and the design, thus unreliable: theory and experimental data very few times agreed! On the other side resonating cavities were offering reliability but at the same time the analysis was complicated by the frequency dependence of the coupling network (narrowband) as was shown by Lynch and York\textsuperscript{xii} and thus not interesting in a practical sense.

York et al results describe the amplitude and phase dynamics with any $N$-port coupling networks and for any oscillator model. The only assumption is the slow variations of the amplitude and phase signals compared to free running carrier frequency and this is for practical purposes satisfied in standard modulation schemes.

When the coupling network satisfy the broadband condition, the equations reduce to previous results found by Stephan\textsuperscript{xi} and York\textsuperscript{xiii} and highlighting their limitations as well as they allow for new design of coupling networks.

Thanks to these results the most convenient coupling schemes in microwave circuits was shown to be based on nearest neighbor interaction and to be easily implemented through transmission lines.

As we will show in the next chapter, this configuration offers a very useful test bench for nonlinear dynamics the exploration of which led to the discovery of the interesting properties we mentioned before, from the easily controlled beam
steering to the noise reduction, from mode-locked pulses to chaotic pattern generation.

In this scenario our research work focused on two aspects.

First we contributed in a better understanding of the dynamics of COAs condensing the information of the discrete set of differential equations into a single partial differential equation in time and space. This equation allows for a simpler calculation of the transient behavior and highlights the limitation associated with size array. Implementation of the discrete and continuous models in Matlab confirmed some of the above-mentioned properties of COAs.

Second we designed COAs using of commercially available oscillators. In the past a major challenge was the design of identical single cells and the extraction of their characteristic parameters used in the mathematical model\textsuperscript{xiv}. Consequently we defined of a set of design rules independent on the particular device used and we emphasized the importance of the correct device characterization and modeling. Furthermore the ability to measure accurately the steady state phase distribution was limited to the analysis of far field radiation patterns. Loading antennas to an oscillator affects its load conditions and introduces mutual radiative coupling between elements. We designed and build a measurement system that overcomes those limitations, providing isolation from the antennas and measuring accurately phase and amplitude at the outputs of the oscillators. The demonstration of its utility was performed with a 2.45 GHz COAs that we designed and built.
**Coupled Phase-Locked Loops Arrays (CPLLAs)**

Although COAs have shown promising results, these are some of their main limitations.

First, most of the interesting results are derived assuming simple models for the oscillators. The researchers trying to include all the parameters of real devices were able to explain some of the observed discrepancy for single injected locked elements, but the extension to a coupled system could be only done numerically.

Second, small differences between unit cells along the array have drastic effects on the array. The intersection of the locking ranges of the single elements determines the overall frequency extension of the synchronized state. In the injection phenomenon the locking range for a single cell is already very limited and thus the whole array has usually a very small range of frequency in which synchronization occurs. Obviously this affects the modulation performances of COAs.

Third, the coupling power and phase information are difficult to control accurately, because at the same microwave frequency of the output signals. Parasitic radiative coupling also usually occurs, making the COAs results sometimes unpredictable.

Finally the phase and amplitude dynamics are coupled together and even if in the future they may lead to some other interesting properties, nowadays the
correlated phase-amplitudes fluctuations have a negative effect on the overall COAs performances.

To obtain the same interesting features of COAs while overcoming these limitations, Martinez and Compton\textsuperscript{xvi} proposed a novel way to implement a nonlinear system able to perform shifterless beamsteering using Coupled Phase-Locked Loops Arrays. Lynch\textsuperscript{xvii} showed that PLLs could be coupled in such a way as to yield essentially to the same phase dynamics of COAs but separated from any amplitude fluctuations and with a larger and controllable locking range. The model behind phase-locked techniques is also well defined, widely studied and relatively independent on the actual devices used in the loop. In addition the coupling occurs at low frequency, so it’s easier to control and design.

On the other side the circuitry becomes more complex and, as we’ll show, new issues related to delays inside the PLL feedback loop need to be faced.

In common with the COAs, CPLLAs share also the way the locking propagates along the array: as shown by the continuous modeling it’s a diffusive type (like heat) so larger arrays take longer time to lock.

In this picture our research work was directed to solve some of the design issues involved in CPLLAs and to prove theoretically and experimentally the advantages of such arrays, such the ability of beamsteering, larger modulation bandwidth and phase noise reduction. Discrete and continuous theoretical model
were implemented and two CPLLAs at 2.45GHz were designed and build to verify the theoretical results.

**Synopsis of the Dissertation**

This dissertation deals with the work undertaken toward a better understanding and implementation of Coupled Oscillator Arrays and Coupled Phase-Locked Loop Arrays. Chapter 2 reviews the dynamics of the single microwave oscillator based on the theoretical accomplishments achieved prior to this research work. Their verification with Matlab simulations will be presented. The presentation will focus mainly on the issues related to the oscillator modeling with the derivation of the amplitude and phase dynamic equations. The steady state, modulation and noise properties will be derived through the analysis of their solutions.

The analysis of the microwave nonlinear oscillator will be extended to an array of interacting elements in Chapter 3. The discrete analysis of the system will be followed by the innovative continuous approach, which confirms the previous results in a more complete scenario.

The important features of COAs, such as beamsteering, modulation capability and noise reduction, will be confirmed by Matlab implementation of the discrete and continuous models in Chapter 4.

Chapter 5 discusses the issues involved with the design, fabrication and characterization of reliable COAs using commercially available components. A
novel phase and amplitude measurement system is then proposed and implemented. The design of two of COAs is presented together with the measurements of their beamsteering, modulation and noise performances.

The theory supporting the interest in CPLLAs, both as discrete and continuous modeling, is presented in Chapter 6, with the theoretical demonstration of beamsteering and faster modulation. The issues related to the delays in the loop and in the coupling schemes will be addressed. Design, fabrication and characterization of CPLLAs will be presented together with the experimental results confirming the advances obtained in the theoretical analysis. Finally summarize and compare the performances and the problems intrinsic to the two systems as we highlight the directions that should be taken to overcome fabrication issues.

Conclusions and suggestions for future research work will end our dissertation.

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CHAPTER 2

Coupled Oscillator Arrays Theory: Single Element Dynamics

The dynamics of Coupled Oscillator Arrays is based on the phenomenon of Injection-Locking (IL). In this chapter we are first going to review the dynamics of the single microwave oscillator based on the theoretical accomplishments achieved prior to this research work. Their verification with Matlab simulations will be presented. The presentation will focus mainly on the issues related to the oscillator modeling with the derivation of the amplitude and phase dynamic equations. The steady state, modulation and noise properties will be derived through the analysis of their solutions.

Injection Locking

The concept of synchronization is extremely important in nature. Our heartbeat is the result of a large number of cells working in synchrony. The movement of worms and snakes is also a result of a coherent synchronization process. When fireflies group together, after few moments the lock occurs: they start flashing with the same interval between light pulses (but with different delays). Same deal with
neurons. And these are just few examples. The several unit cells that compose the system communicate each other (injection) till they reach a common regime characterized by the same time properties. An external signal can also affect by injection the dynamical behavior of the whole system.

As deeply explored by Van der Pol\textsuperscript{iv} in the early twentieth century, the dynamics of nonlinear oscillating electrical circuits is affected by the injection of an external signal. This effect has a long history dating back to observations of synchronized mechanical pendulums by Huygens\textsuperscript{v}, who was the first to describe it mathematically. Several behavioral changes have being observed, but the case of most interest occurs when the period of the oscillations becomes the same of the external source and thus the oscillator’s frequency is locked to the injection signal. In this case the injection is unilateral (Figure 1a), because the oscillator does not affect the injection signal. When the injection is bilateral (or mutual), then the sources influence each other and synchronization occurs when they reach a common frequency (Figure 1b).
Figure 1: Unilateral (a) and bilateral (b) injection of nonlinear oscillators.

The interesting aspect of the locking state of the injection phenomenon (in both mutual and unilateral cases) is that the phase difference between the two elements is directly related to the original free running (or natural) frequencies, following the Adler’s equation:\(^\dagger\):

\[
\frac{d\phi}{dt} = \omega_o - \omega_{inj} + \Delta\omega_m \sin(\psi - \phi) \tag{2.1}
\]

where, as we will define later, \( \Delta\omega_m = \frac{\omega_o \rho}{2Q A} \) is the locking range (\(Q\) is the quality factor of the oscillator) and the other quantities are defined in Figure 2a.
In steady-state (when \( \frac{d\phi}{dt} = 0 \)), the phase difference satisfies:

\[
\psi - \phi = \sin^{-1}\left(\frac{\omega_{\text{inj}} - \omega_o}{\Delta\omega_m}\right)
\]

Thus, the phase difference between injected and output signal can be controlled through the free running frequency of the oscillator within the locking range. It is then clear that the most convenient oscillating device for microwave COAs is a Voltage Controlled Oscillator (VCO) as the tuning port allows for easy voltage
control of the free running (or natural) frequency. When two identical sources are used in a mutual coupling scheme, then simple frequency detuning can control the phase difference between the two outputs. The coherent addition of the synchronized and diphased signals in space gives rise to low loss power combining and beam forming. Simple detuning causes beamsteering as brightly proposed Stephan vii. For systems with more than two elements, several ideas have been proposed, but the only ones that can be efficiently implemented at microwave (and millimeter-wave) frequencies are the ones proposed by Stephan viii and York ix. To understand their schemes and to derive the modulation and noise performance of COAs, we need to extend the theory of nonlinear oscillations to network of microwave oscillators coupled by mutual injection and ultimately locked to an external reference source. Several methods are available in the literature, but the approach used by Kurokawa x, based deeply on network concepts, gives substantial engineering insight into the behavior of COAs. The method was first used to describe modes of operation and noise performance of microwave nonlinear oscillators. As we will show, starting from frequency-domain description of the oscillator and the coupling network, a rigorous expansion of the of the time-domain response is derived through the inverse Fourier transform. The analysis presented here remains faithful to the comprehensive work of York xi. We start with the derivation of the dynamics properties of the unit cell of the array, the nonlinear microwave oscillator. Then we will extend the study to the array of oscillators coupled through a generic N-Port network. The theory will address the
amplitude/phase dynamics, the stability of steady-state solutions, the beamsteering transient, the amplitude/frequency/phase modulation issues and the noise features.

**Dynamical Model**

The model associated to microwave can be extremely complicated based on the active device and the embedding circuitry used. However experience shows that fruitful insight of the nonlinear dynamics can be usually obtained with a simple model where the linear contribution is frequency dependent while the nonlinear part is only function of the amplitude. Thus the linear reactive part includes the effects of the embedding passive circuit and the device parasitics, as the nonlinearity is associated to the negative resistance (power generating) active device. In reality most of the actual devices, such as the MESFETs, have also a nonlinear reactive part, but even if the analysis can be extended to such cases\textsuperscript{xii}, here we will make our first assumption using the simplest parallel model shown in Figure 3a. An equivalent serial model can be used and with the appropriate modifications it leads to analog results, as we will show later\textsuperscript{xiii}. The injection of an external signal can be functionally represented by an independent current source $I_{\text{inj}}$. From Kirchoff's Current Law we can obtain the starting relationship:

$$\tilde{I}_{\text{inj}}(\omega) = \tilde{V}(\omega)Y_{t}(\omega,|V|)$$  \hspace{1cm} (2.3)
where the tilde (~) indicates a frequency domain (phasor) quantity and for later notation convenience we have identified a “total admittance” across the output terminals as

$$Y_t(\omega, |V|) = \left[ Y_{osc}(\omega, |V|) + Y_L(\omega) \right]$$  \hspace{1cm} (2.4)

Most practical oscillators are designed to work in single-mode operation, to avoid frequency jumps, uncontrolled power fluctuations and strong dependency on loading conditions. This justifies our second hypothesis of a single resonant circuit with high $Q$ factor to sustain nearly sinusoidal oscillations around a nominal center frequency $\omega_0$, characteristic of the unperturbed system, thus $\omega \approx \omega_0$. Being the $Q$ large, the bandwidth of the amplitude and phase variations is much smaller than the carrier. In typical communication systems this postulation is clearly valid.

As a consequence the system can be modeled with a single resonant LC tank, a resistive load and the amplitude dependent negative conductance as shown in Figure 3b.

![General parallel model of oscillator (a) and single-mode implementation (b).](image-url)
This assumption is valid if the resonance is well defined and isolated
\( \left( \omega_o = 1/\sqrt{LC} \right) \) with high \( Q \) factor (usually > 10). A variable capacitor
(incorporated in \( C \)) can then provide the necessary tuning.

Starting from these simplifying assumptions, we can derive the system’s
dynamic equations transforming the circuit equations from the frequency domain to
the time domain.

Because the system is designed to produce nearly sinusoidal oscillations around
the free running frequency \( \omega_o \) of the resonant tank, the time-dependent output
temperature can be written in the useful complex form:

\[
V(t) = A(t)e^{j(\omega_o t + \phi(t))} = V'(t)e^{j\omega_r t}
\]

(2.5)

where \( A \) and \( \phi \) are the dynamic amplitude and phase variables,
\( V'(t) = A(t)e^{j\phi(t)} \) is the output phasor and \( \omega_r \) is a reference frequency close to \( \omega_o \).

By hypothesis the phase and amplitude variations are slow compared to the carrier
and thus defining \( V'(t) \) and \( \omega_r \) will show its utility in the next passages as they
describe the perturbation of the system from its free running state.

Applying the inverse Fourier transform to (2.3) and exploiting the slow-varying
assumptions \( \frac{d\phi}{dt} \ll \omega_o, \frac{1}{A} \frac{dA}{dt} \ll \omega_o \), it can be shown (“Kurokawa substitution” in
Appendix A) that (2.3) becomes
\[ I_{\text{inj}}(t) = V(t) \left[ Y_t(\omega_r, A) + \frac{\partial Y_t(\omega_r, A)}{\partial \omega} \left( \frac{\partial \phi}{\partial t} - j \frac{1}{A} \frac{\partial A}{\partial t} \right) \right] \]  \hspace{1cm} (2.6)

Separating the real and imaginary parts, we can solve for the amplitude and phase variations

\[ \frac{dA}{dt} = A \Re \{ F(A, \phi) \} \]  \hspace{1cm} (2.7)

\[ \frac{d\phi}{dt} = A \Im \{ F(A, \phi) \} \]  \hspace{1cm} (2.8)

with

\[ F(A, \phi) = \frac{I_{\text{inj}} - Y_t(\omega_r, A)}{V} \frac{\partial Y_t(\omega_r, A)}{\partial (j\omega)} \]  \hspace{1cm} (2.9)

We can cast (2.7) and (2.8) in a standard form for nonlinear analysis recalling that \( V'(t) = A(t)e^{j\phi(t)} \) and thus \( F(V') = F(A, \phi) \):

\[ \frac{dV'}{dt} = F(V')V' \]  \hspace{1cm} (2.10)

As described in Appendix A (2.6) is valid when higher-order derivatives of amplitude and phase, as well as higher-order frequency derivatives of the total admittance, are negligible compared with the first two terms of the Taylor expansion (the terms in the square brackets in (2.6)). As we previously stated, this is the case of slowly varying modulation and/or broadband embedding network,
excellent practical assumption. We will see that the ability of modulate COAs is directly related to the validity of this assumption.

To define $V'(t)$ we need to specify a particular circuit and the one in Figure 3b showed to be good enough to explain most of the experimental observations. Its total admittance can be easily calculated, evaluated at the reference frequency $\omega_r$ and put in the convenient form

$$Y_i(\omega_r, A) = G_d(A) + G_L + 2jC(\omega_r - \omega_o) \quad (2.11)$$

Thus the (2.9) becomes

$$F(A, \phi) = \frac{I_{inj}}{2CV} - \frac{1}{2C} \left[ G_d(A) + G_L \right] + j(\omega_o - \omega_r) \quad (2.12)$$

At microwave frequencies it is most convenient to express the dynamics equations in terms of easily measurable quantities as resonant frequency and $Q$ factor. This last for a parallel resonant tank can be defined as $Q = \omega_r C / G_L$.

Moreover the injection signal can be conveniently expressed as $I_{inj} = \rho G_L e^{i[\omega_o t + \psi(t) - \phi]}$. Recalling that $V = A(t) e^{i[\omega_o t + \psi(t)]}$, the (2.12) can be cast as

$$F(A, \phi) = \frac{\omega_o}{2Q} A e^{i[\omega_o - \omega_r) t + \psi(t) - \phi]} - \frac{\omega_o}{2Q} \left[ \frac{G_d(A)}{G_L} + 1 \right] + j(\omega_o - \omega_r) \quad (2.13)$$

which leads to the important dynamics equations

$$\frac{dA}{dt} = \frac{\omega_o}{2Q} \rho \cos \left[ (\omega_{inj} - \omega_r) t + (\psi - \phi) \right] - \frac{\omega_o}{2Q} A \left[ \frac{G_d(A)}{G_L} + 1 \right] \quad (2.14)$$
\[
\frac{d\phi}{dt} = \frac{\omega_o}{2Q} A \rho \sin \left[ \left( \omega_{inj} - \omega_r \right) t + (\psi - \phi) \right] + (\omega_o - \omega_r) \tag{2.15}
\]

Before starting the study of the solutions, it is important to note that the reference frequency \( \omega_r \) is not \( \omega_{inj} \) or \( \omega_o \) and will be chosen for convenience for each particular situation: establishing an opportune harmonic reference for the time variation of \( \phi \) will simplify the calculation and understanding of the phase dynamics.

**Free-Running Oscillator**

When no injection signal is present, the governing equations reduce to

\[
\frac{dA}{dt} = -\frac{\omega_o}{2Q} A \left[ \frac{G_d(A)}{G_L} + 1 \right] \tag{2.16}
\]

\[
\frac{d\phi}{dt} = \omega_o - \omega_r \tag{2.17}
\]

Let’s start from the steady-state analysis. When \( \frac{d}{dt} = 0 \) the system will oscillate at the natural frequency \( (\omega_r = \omega_o) \), while the amplitude will satisfy the equation

\[
G_d(A) + G_L = 0 \tag{2.18}
\]

The functional dependence of \( G_d \) from \( A \) depends on the particular oscillator used and its derivation for a particular model, such as the Meissner \( ^{xiv} \) or the Van der Pol \( ^{xv} \), can be found in the literature. All the studies make use of the averaging
method used in classical nonlinear theory\textsuperscript{xvi} to obtain a polynomial expansion of the conductance in powers of \( A \). The following analysis will be based on the Van der Pol model, but it can be extended to other models. However, the phase behavior is far more interesting in practical applications (as beamsteering, modulation and noise,) than the amplitude one, so our final goal is actually to reduce the influence of the amplitude on the overall dynamics. Thus the particular model, appearing only in the amplitude equation, does not profoundly affect the understanding of the system behavior.

Following Van der Pol we can write

\begin{equation}
G_d(A) = -G_0 + G_2 A^2 \tag{2.19}
\end{equation}

where \( G_0 \) is the small-signal negative conductance and \( G_2 \) describes the nonlinear quadratic dependence. The quantities in (2.19) are not easily measurable. To find a more convenient expression, let’s rewrite (2.16) using (2.19):

\begin{equation}
\frac{dA}{dt} = \frac{\omega_0}{2Q} \left[ \left( \frac{G_0}{G_L} - 1 \right) - \frac{G_2}{G_L} A^2 \right] \tag{2.20}
\end{equation}

and rearranging we obtain

\begin{equation}
\frac{dA}{dt} = \frac{\omega_0}{2Q} \left( \frac{G_0}{G_L} - \frac{G_2}{G_L} \right) \left[ 1 - \frac{G_2}{G_0 - G_L} A^2 \right] \tag{2.21}
\end{equation}

It is clear that in steady-state the amplitude of the oscillations will satisfy
\[ A = \sqrt{\frac{G_0 - G_L}{G_2}} \triangleq \alpha \]  
(2.22)

and the strength of the perturbed behavior is related to the ‘nonlinearity parameter’ defined as

\[ \mu \triangleq \frac{G_0 - G_L}{G_L} \]  
(2.23)

These newly defined quantities are measurable as \( \alpha \) is the free-running amplitude and \( \mu \) can be measured from the transient response of the oscillator. Recalling the freedom in the preference of \( \omega_r \), in this particular case, we can find useful to define it as \( \omega_r = \omega_0 \). These preferences reduces (2.16) and (2.17) to simpler and more useful expressions:

\[ \frac{dA}{dt} = \mu \frac{\omega_0}{2Q} A \left(1 - \frac{A^2}{\alpha^2}\right) \]  
(2.24)

\[ \frac{d\phi}{dt} = 0 \]  
(2.25)

The two equations are uncoupled. Three solutions for the steady-state can be found imposing \( dA/dt = 0 \), and they are \( A = \hat{A} = 0, +\alpha, -\alpha \), where \( \hat{A} \) denotes any of the solution. Because \( A \geq 0 \) (the negative sign can be included in the phase information as a \( \pi \) shift), let’s check the stability of the other two solutions, using a perturbation analysis: \( A = \hat{A} + \delta A \)
\[
\frac{d(\dot{A} + \delta A)}{dt} = \frac{d(\delta A)}{dt} \approx f(\dot{A}) + f'(\dot{A})\delta A = \frac{\omega_0}{2Q} \left[ \frac{\dot{A}}{\alpha} - \frac{\dot{A}^2}{\alpha^2} \right] + \frac{\omega_0}{2Q} \left[ 1 - \frac{3\dot{A}^2}{\alpha^2} \right] \delta A
\]

(2.26)

where \( f \) indicates the generic function of the amplitude. Therefore

\[
\frac{d(\delta A)}{dt} = \begin{cases} 
\mu \frac{\omega_0}{2Q} \delta A & \text{for } \dot{A} = 0 \\
-\mu \frac{\omega_0}{2Q} \delta A & \text{for } \dot{A} = \alpha 
\end{cases}
\]

(2.27)

\( \mu > 0 \) will make sure that the amplitude diverges if a perturbation is applied around the zero solution, while around \( \alpha \) is stable. Using (2.23) and (2.22) we find the startup requirement for a parallel oscillator and the Kurokawa’s stability condition \( dG_L/dA > 0 \):

\[
\begin{align*}
\mu > 0 & \quad G_0 > G_L \\
\alpha > 0 & \quad G_0 > G_L
\end{align*}
\]

(2.28)

An interesting way to study the dynamics is to plot the phase diagram for the startup of the oscillation. The system reaches the steady-state amplitude after a transient determined by the value of \( \mu \).

To demonstrate this idea, we implemented in Matlab® the (2.24) and (2.25) as a function of the nonlinear parameter. In the Simulink® Package we can create nonlinear dynamical systems using basic mathematical functional blocks, such as
integrators and polynomials. Their solution is calculated in time domain using very efficient computational methods such as the Dormand-Prince and the Bogacki-Shampine.

![Diagram of Matlab implementation of the Van der Pol nonlinear oscillator model.](image)

Figure 4: Matlab implementation of the Van der Pol nonlinear oscillator model.

To understand the model used in Simulink® we need to cast (2.24) in a way compatible with the block diagram shown in Figure 4:

\[
\frac{d}{dt} \left( \frac{A}{\alpha} \right) = \mu \frac{\pi}{Q} \left( F_o T \right) A \left[ 1 - \left( \frac{A}{\alpha} \right)^2 \right] \rightarrow \frac{dx}{d\tau} = b \left( x - x^3 \right)
\]

(2.29)

where for simplicity we defined the quantities

\[
\tau = \frac{t}{T}, \quad x = \frac{A}{\alpha}, \quad f_o = \frac{F_o}{F} = F_o T \quad \text{and} \quad b = \mu \frac{\pi}{Q} f_o
\]

(2.30)
Concerning the frequency of the oscillation, it is clearly $\omega_0$, but as it will be clear later, it is convenient to define the total phase and the instantaneous frequency as

$$\theta(t) = \omega_0 t + \varphi(t)$$
$$\theta'(t) = \omega_0 + \varphi'(t)$$  \hspace{1cm} (2.31)

Visibly the total phase is obtained as the integral of the frequency and this explains the integrator followed by the sinusoidal function in Figure 4. There is no input for phase and amplitude because the system is in its free-running state. The only source is an initial small perturbation ($10^{-9}$ time smaller than the steady-state amplitude) to start the oscillation. In real systems this perturbation models bias startup or typical noise. Being the phase a relative quantity, we set the initial value to zero. Note that we defined

$$V_{out} = A(t) \cos[\theta(t)] = \Re \left\{ A(t) e^{i\theta(t)} \right\} = \Re \left\{ V(t) \right\}$$  \hspace{1cm} (2.32)

The results of our simulations for $\mu = 3, 6$ and 12 are in Figure 5 and Figure 6.

We assumed the following values based on a typical oscillator used for our arrays

$$\alpha = 1 \text{ V}$$
$$T = 1 \text{ ns} \rightarrow F = 1 \text{ GHz}$$
$$f_o = 2.45 \text{ and } Q = 10 \rightarrow b = 0.771\mu$$  \hspace{1cm} (2.33)
An interesting way to view the VDP response as function of the nonlinearity is the phase plot of the response in time and its derivative (Figure 6). For a sinusoidal output, a circle in the phase plane represents the steady state. The signal is bounded by the $\pm \alpha = \pm 1 \text{ V}$, while it’s derivative limits are $\pm 2\pi f_o \alpha = \pm 15.4 \text{ V/ns}$. To reach this stable state starting from the origin (initial condition $V_{\text{out}}(0) = 0$ and $V'_{\text{out}}(0) = 0$), the system follows a shorter trajectory as stronger is the nonlinearity. A linear system will not able to oscillate without an oscillating input. Clearly the fastest transient is associated to larger $\mu$ as also shown in Figure 5.
A last point before the study of the injection-locked oscillator: from (2.30) to maintain the same behavior, at higher frequency we need higher $Q$ and thus the implementation becomes more challenging at microwaves.

**Injection-Locked Oscillator**

When an external signal is present, than (2.14) and (2.15) need to be used. Also in this case the arbitrariness of $\omega_r$ comes useful: it is convenient to chose $\omega_r = \omega_{\text{inj}}$ to eliminate the explicit time dependence on the phase equation. Using this reference frequency and using the VDP model, the dynamic equations become

$$\frac{dA}{dt} = \frac{\omega_0}{2Q} \rho \cos(\psi - \phi) + \mu \frac{\omega_0}{2Q} A \left( 1 - \frac{A^2}{\alpha^2} \right)$$  \hspace{1cm} (2.34)
\[
\frac{d\phi}{dt} = \omega_o - \omega_{\text{inj}} + \frac{\omega_o \rho}{2Q A} \sin(\psi - \phi)
\] (2.35)

Evidently the two equations are more uncoupled as the injection is smaller \((\rho \approx 0)\). In that case the steady-state amplitude is close to \(\alpha\) and the behavior is close to the free running oscillator.

When the amplitude is close to a constant value (small fluctuations \(\rightarrow dA/dt \approx 0\)), the phase evolution (2.35) is clearly governed by the Adler’s equation (2.1) mentioned before. Visibly this happens in two cases: close to steady-state or for very small nonlinearity \(\mu\) and injection strength \(\rho\). It’s important to note that we arrived to this expression without involving any particular property of the VDP model, present in the amplitude dynamics. Thus it is a result of a more general character.

In the steady state (or small fluctuations in general) the three solutions can be obtained in a simple way by recasting (2.34) and defining \(\beta = \frac{\rho}{\mu} \cos(\psi - \phi)\)

\[
\frac{dA}{dt} \approx 0 \rightarrow \beta \approx \frac{A^3}{\alpha^2} - A
\] (2.36)

As shown in Figure 7, the real solutions are the intersection points between a constant value of \(\beta\) and the 3rd order polynomial curve. The plot can be rotated to highlight that the real solution bifurcates into two then three than two again for increasing \(\beta\). In effect, only a positive value of \(\beta\) is interesting, because, as we will
show soon, the phase stable points are from $-\pi/2$ to $+\pi/2$. Also $\alpha$ must be non-negative as we defined before. Thus the intersection points in the yellow quadrant are the interesting ones and they are described by the same branch equation (Sol₃), as we calculated with Mathematica® (Appendix B1)

$$A = \tilde{\alpha} \left( \frac{1}{\sqrt[3]{\gamma} + \sqrt[3]{\gamma^2 - 1}} \right) + \sqrt[3]{\gamma + \sqrt[3]{\gamma^2 - 1}} \right) \text{ with } \gamma = \frac{3\beta}{2\tilde{\alpha}} \text{ and } \tilde{\alpha} = \frac{\alpha}{\sqrt{3}} \quad (2.37)$$

Apart from the case $\beta = 0$, which we analyzed previously, only one real solution is present in the yellow quadrant and three situations may be highlighted. When $\beta \geq \frac{2\alpha}{\sqrt{3}}$ the solution is clearly real. Developing in the complex domain, it can be shown (also Appendix B1), that also in the range $0 < \beta < \frac{2\alpha}{\sqrt{3}}$ the solution is real.
As evident in the figure and proved in Appendix B2, the amplitude is always greater than the free-running steady state value and it increases as the injected strength rises, since the derivative of $A$ versus $\beta$ is always positive (Appendix B3).

As third case, it is interesting to see the variation of the amplitude for small injection signal. Developing the Taylor expansion for $\beta \to 0$ (Appendix B3), we can write

$$A \approx \alpha + \frac{1}{2} \beta - \frac{3}{8 \alpha} \beta^2 + O(\beta^3)$$ \hspace{1cm} (2.38)

This gives us an idea of the amplitude change for small perturbations from its free running situation. Recalling that $|\beta| \leq \rho/\mu$, the injection strength needs to be small relatively to the nonlinearity: for strongly nonlinear system the perturbation does not affect the system, which tries to stick to its natural (free-running) state.

Concerning the phase, the steady-state reveals a very interesting feature that will be later exploited for the beamsteering technique. From (2.35) we can write

$$\Delta \phi = \psi - \phi = \sin^{-1} \left( \frac{\omega_{inj} - \omega_0}{\Delta \omega_m} \right) \text{ with } \Delta \omega_m = \frac{\omega_0 P}{2Q A}$$ \hspace{1cm} (2.39)
Assuming small amplitude fluctuations and injection strength constant, the output signal is synchronized to the injected signal only if this lies within the locking range of the oscillator, $\omega_o \pm \Delta \omega_m$. Now two situations may occur.

If we keep constant the natural frequency and vary the injection one, the locking range is constant for $\omega_{inj}$. However the output frequency changes from $\omega_o - \Delta \omega_m$ to $\omega_o + \Delta \omega_m$ while the phase varies from $-\pi/2$ to $\pi/2$, since the signals are locked.

On the other side if we want to control the phase shift by changing the free-running frequency, than the locking range is not symmetric anymore.

\[
\omega_{inj}\left[1 - \frac{\Delta \omega_m}{\omega_o} + \left(\frac{\Delta \omega_m}{\omega_o}\right)^2\right] \approx \frac{\omega_{inj}}{1 + \frac{\Delta \omega_m}{\omega_o}} \leq \omega_o \leq \frac{\omega_{inj}}{1 + \frac{\Delta \omega_m}{\omega_o}} \approx \omega_{inj}\left[1 + \frac{\Delta \omega_m}{\omega_o} + \left(\frac{\Delta \omega_m}{\omega_o}\right)^2\right]
\]

(2.40)

Nonetheless, in first approximation if $\Delta \omega_m \ll \omega_o$ and $\omega_{inj} \ll \omega_o$ then the difference between the two situations is negligible. The phase this time decrease from $\pi/2$ to $-\pi/2$. The curves associated with the two types of phase control are shown in Figure 8 for typical parameters’ values.

In both cases if a large locking range is desired, low-$Q$ oscillators with large injection strengths are required.
Concerning this phase shifting capability, a last issue needs to be addressed. As we know, the inverse sine function gives two possible solutions for the phase difference in this range; which of the two is stable is determined by the stability analysis as we did for the amplitude solutions. A small perturbation from the steady state $\phi = \hat{\phi} + \delta \phi$ reduces (2.35) to

$$\frac{d}{dt}(\hat{\phi} + \delta \phi) = \frac{d}{dt}(\delta \phi) \approx f(\hat{\phi}) + f'(\hat{\phi})\delta \phi =$$

$$= \omega_o - \omega_{inj} + \Delta \omega_m \sin(\psi - \hat{\phi}) - \Delta \omega_m \cos(\psi - \hat{\phi}) \delta \phi$$

Thus the perturbation will decay in time provided that

$$\cos(\psi - \hat{\phi}) \geq 0 \rightarrow |\psi - \hat{\phi}| \leq \pi/2$$
This result explains the evolution of the stable phase within the locking range in Figure 8.

The injection-locked system can also be modeled in Matlab: again we cast the equations in a way convenient for the implementation using the definitions (2.30) to which we add the injection coefficient \( c = \frac{\rho \pi}{\alpha Q} f_0 \), which express the ratio between the injection signal strength and the free-running amplitude.

\[
\begin{align*}
\frac{dx}{d\tau} &= b(x-x^3) + c \cos(\psi-\phi) \\
\frac{d\phi}{d\tau} &= \omega_o - \omega_{inj} + \frac{c}{x} \sin(\psi-\phi)
\end{align*}
\tag{2.43}
\tag{2.44}
\]

For later convenience, we will assume that \( \psi = \psi(t) \) and \( \rho = \rho(t) \) to be able to simulate modulation of the external source. However the amplitude modulation is not so interesting from a practical point of view since it would require a complete knowledge of the nonlinear amplitude characteristic and, as we said previously, each device has different model. The model schematic is shown in Figure 9.
Figure 9: Matlab implementation of the injection-locked VDP nonlinear oscillator model.

To have an idea of the typical response of the VDP oscillator (VDPO) to the injection, we simulated a 2 V amplitude step (AM) followed by a 90° phase step (PM). The amplitude and phase are modulated after the oscillator has reached the free-running state. Figure 10 confirms what previously stated: the injection affects the steady-state amplitude and the coupling between the equations explains the amplitude glitch when the phase step is applied.

Note two details. In the AM region $\beta = \frac{\rho}{\mu} \cos(\psi - \phi) = \frac{2}{6} \cos(0)$ and plugging this value in (2.37) gives steady-state amplitude of 1.137 V as in Figure 10. In the PM region, the oscillator sees no initial injection in the amplitude because $\cos(\pi/2) = 0$, thus it tends to go to its free running state, but the coupling
between phase and amplitude equations force the phase difference to be reduced and this explains qualitatively the glitch.

Concerning the frequency modulation, a change in $\omega_{\text{inj}}$ will affect the steady-state frequency (because of the lock) but also the phase as Adler’s equation indicates and the amplitude as (2.43) points out. So, as we will soon show, the FM is not a good modulation scheme from the practical point of view.

With a VCO the manipulation of $\omega_o$ can be easily implemented. Such control allows for simple PM without affecting the output frequency: the difference between the output and injected signals is described by (2.39). The downside is that also the amplitude and the locking range are affected by such modulation scheme. In the next section this tradeoff between amplitude fluctuations and locking range will be presented with greater details.
Modulation

To transmit an information signal over the air, there are three main steps. First a pure carrier is generated at the transmitter. The carrier is modulated with the information to be transmitted. Any reliably detectable change in signal characteristics can carry information. At the receiver the signal modifications or changes are detected and demodulated.

Over the past few years a major transition has occurred from simple analog Amplitude Modulation (AM) and Frequency/Phase Modulation (FM/PM) to new digital modulation techniques. Examples of digital modulation include QPSK (Quadrature Phase Shift Keying), FSK (Frequency Shift Keying) and QAM (Quadrature Amplitude Modulation).

Then another layer of complexity was added with many new multiplexing systems. Two principal types of multiplexing (or “multiple access”) are TDMA (Time Division Multiple Access) and CDMA (Code Division Multiple Access). These are two different ways to add diversity to signals allowing different signals to be separated from one another (Figure 11a).
Independently on the particular technique, there are only three characteristics of a signal that can be changed over time: amplitude, phase, or frequency (Figure 11b). Furthermore phase and frequency are just different ways to view or measure the same signal change.

In AM, the amplitude of a high-frequency carrier signal is varied in proportion to the instantaneous amplitude of the modulating message signal.

Frequency Modulation (FM) is the most popular analog modulation technique used in mobile communications systems. In FM, the amplitude of the modulating
carrier is kept constant while its frequency is varied by the modulating message signal.

Amplitude and phase can be modulated simultaneously and separately, but this is difficult to generate, and especially difficult to detect. Instead, in practical systems the signal is separated into another set of independent components: in-phase (I) and quadrature (Q). These components are orthogonal and do not interfere with each other. This allows for digital schemes to be implemented in a practically convenient way\textsuperscript{xix}.

Here we are interested only in the three standard basic scalar techniques to avoid circuit complexity and focus only in the main features of COAs (Figure 11a,b).

**Amplitude Modulation**

The easiest way to implement an amplitude modulation would be to add a controlled attenuator at the output of the oscillator. From one side this would avoid the need for an accurate knowledge of the amplitude dynamics, but would add circuit complexity and for larger number of oscillators the cost would become prohibitive.

Another way is to modulate the injection source and as previously showed the amplitude of the oscillator output changes following (2.37) if the modulation rate is small enough to consider the system close to steady state. To verify this hypothesis this modulation scheme was implemented in the Matlab model.
Figure 12: AM response as function of the nonlinear parameter (a) and the injection strength (b).

Three factors play an essential role in the oscillator response to AM: the injection amplitude $\rho$ (modulation offset and span), the nonlinear parameter $\mu$ and the phase difference between injected and oscillator signals.

When the stationary phase difference $\Delta\phi$ is zero the amplitude behavior is decoupled from the phase and the (2.43) reduces to

$$\frac{dx}{d\tau} = \mu \frac{x}{Q} \left( x^3 \right) + \frac{\rho \pi}{\alpha Q} \frac{f_o}{c}$$

(2.45)
Thus, when $\omega_o = \omega_{inj}$, $\mu$ and $\rho/\alpha$ are the only two parameters affecting the modulation as clearly visible in Figure 12. Unfortunately precise control of the output amplitude requires accurate knowledge of the nonlinearity function and its parameters. If the VDP model is accurate for the oscillator used, then the knowledge of $\mu$ is enough, but this is not always the case. Moreover the amplitude value is the solution of (2.37): it needs to be calculated numerically and this is not desirable in practical systems.

Figure 13: Phase plane for the AM: starting from $\rho_{\text{offset}}$, the injection steps from $\rho_{\text{min}}$ to $\rho_{\text{max}}$.

An interesting way to look at the AM response is through the phase plane previously mentioned. The stable states are three and they relate to the different values of the injection strength associated with the AM: the offset references the modulation around a certain value $\rho_{\text{offset}}$, while the modulation sweeps between $\rho_{\text{min}}$ and $\rho_{\text{max}}$. The system will thus follow mainly three circular trajectories.
corresponding to the amplitude values reached after each transient, which in turn are represented by lines joining the steady state solutions.

**Frequency Modulation**

The frequency modulation presents few challenges. First simply changing $\omega_0$ would not affect the output frequency because the oscillator is locked to the external source. Second varying only the injection frequency would modulate the oscillator frequency, but also the relative phase difference as clear from (2.39). This in turn causes AM because the steady state amplitude depends on the phase difference as in (2.37). We implemented the frequency modulation of the injected source in our model and the results are shown in Figure 14 for modulation span of 50, 100 and 122.5 MHz around the carrier of 2.45 GHz. The last value corresponds to the edge of the locking range, using the values $A = \alpha = \rho = 1$, $\mu = 6$ and $Q = 10$. The frequency span is the maximum possible before reaching the phase instability close to 90° mentioned before.

Clearly amplitude and stationary phase difference are also modulated, but the amplitude fluctuation is not as significant as the phase one. Plus increasing the modulation span increases the phase difference from 0 to 90°, but reduces the amplitude fluctuations as the two events are associated to sinus and cosine functions respectively:

\[
|\omega_0 - \omega_{inj}| \xrightarrow{\text{Phase Dynamics}} \sin(\psi - \phi) \xrightarrow{\text{Amplitude Dynamics}} \cos(\psi - \phi) \Rightarrow A \rightarrow \alpha \quad (2.46)
\]
Moreover frequency, amplitude and phase have a common transient before reaching the desired modulation state. The length of this transient and the intensity of the unwanted fluctuations are directly related to the frequency span as the instantaneous $\omega_{\text{inj}} - \omega_o$ causes the formation of a phase difference, which therefore force the amplitude to change (visibly from (2.35) followed by (2.34)).

To overcome these problems, injection source and natural frequency need to be modulated simultaneously. The complexity of triggering the two modulations is the practical drawback of such idea. However, an additional advantage emerges: theoretically the locking range does not limit anymore the modulation span, since $\omega_{\text{inj}} = \omega_o$ and thus the phase is constant within the tuning range, defined as the frequency interval of $\omega_o$ sweep. The tuning range is usually much wider that the locking range in injection locked oscillators (we will see that this is not always true for phase-locked loops).
Phase Modulation

Phase modulation seems to be the most useful in a practical sense. Simple control of the natural frequency within the locking range (with the small modification in) (2.40) leads to variation of the phase within ±90°. Also modulating directly the injection source would lead the output oscillator to change its phase accordingly (Figure 15). The difference between the two methods resides in the fact that the phase difference between injection and oscillator varies with \( \omega_0 \), while it
remains constant when $\omega_{\text{inj}}$ is modified. As we will see, for beamsteering a stationary phase difference between the two signal sources is needed. Thus for scanning, adjusting $\omega_o$ would be the best choice, while for sending phase modulated signal – once the direction of the beam is set – control of $\omega_{\text{inj}}$ is recommended.

As evident from (2.40), the control of the natural frequency is asymmetric, as also the locking range changes consequently (Figure 15a and Figure 16b). Amplitude and output frequency are also affected by the phase adjustment for both

![Figure 15: Output phase (a) and phase difference (b) for the PM response to $f_o$ or $\nu$ control.](image)
\(\psi\) and \(F_o\) controls since the phase difference between injection and oscillator changes during modulation. However, their final values after the transient do not change. As for FM, the transient is longer as the modulation span, increases toward its limit of \(\pi/2\) (Figure 16a).

![Figure 16](image_url)

Figure 16: Response to various PM spans (a). Amplitude and frequency changes due to PM (b).

**Time Delay Considerations**

The modulations schemes here proposed, no true time delay (also referred as transport delay) was considered to occur along the signal propagation in the oscillator. The reason is that the analysis becomes extremely complicated and only
numerical simulations can be performed to obtain a qualitative idea of the behavioral changes. However, as we will also see for phase-locked loops, the effect of this delay is clearly visible in real devices and in Matlab models.

Also the effect of delay between oscillators in COAs and phase-locked loops in CPLAs has been addressed in the literature and it will be analyzed in details later.

Now we just give a qualitative insight. The addition of delays in the governing equations introduces instability and bifurcation in the solution set. The desired stable solution can still be reach within new (and smaller) frequency and amplitude ranges.

In both the single element and the array the effect of such delays is critical for determining the modulation bandwidth: the acquisition time and thus the maximum modulation rate relate to the ability of the system to ‘capture’ (pull-in) an injected signal and delays hinder this process.

On the other side, the locking range is not affected, since it is a quantity defined in steady state condition (it can be seen as a very slow modulation within the previously mentioned limits).

Therefore the previously mentioned schemes are still valid, but the time scales should be considered only qualitatively correct, as the system transient would last longer as delays in the real circuitry slow down the locking course of action.
As an example we modify (2.43) and (2.44) to include only a delay $T_d$ in the phase loop (see Figure 17)

$$\frac{d\phi(\tau)}{d\tau} = \omega_o - \omega_{inj} + \frac{c}{x} \sin[\psi - \phi(\tau - T_d)]$$ (2.47)

Assuming the same realistic parameters in Figure 10, the response to an amplitude and phase steps is shown in for several values of the transport delay. A 10 MHz difference between $\omega_o$ and $\omega_{inj}$ is used to highlight the effect of the delay in (2.48): this explains the phase linear variation with time before lock.
Figure 18: Oscillator response to external injection steps for several values of the intrinsic delay: injection signal evolution (a) and VDPO transient for $T_d = 408, 612$ and $816$ ps (b).

Clearly a small delay compared to the oscillation period does not affect significantly the transient response, while a larger delay severely limits the modulation bandwidth.

**Noise**

The noise performance of a nonlinear oscillator can be drastically improved through the mechanism of injection locking to a low noise source. As a matter of fact the oscillator locks to an external source and follows the noise features of the external injection. As we will see this phenomenon explains the noise reduction in COAs both in the free-running state and when an external reference source is used.

Noise can be seen as random fluctuations in the values of amplitude and/or phase (thus frequency also). Usually phase noise is the most relevant for oscillators and it occurs naturally in electronic circuits. It can be observed in the time domain
as phase jitter of the signal on an oscilloscope display or as time fluctuations of the zero crossings (Figure 19a). Power supplies are usually the main sources of such noise, since high quality circuitry biasing provides less noisy oscillators.

\[ V_o = \cos(\omega t + \phi) \]

\[ \Delta \phi \leftrightarrow \Delta t \]

**Figure 19:** Phase noise represented in the time (a) and in the frequency (b) domains.

Phase noise and frequency fluctuations are the same physical phenomenon. Noise in angular frequency can be obtained from the derivative of phase with respect to time.

\[
f(t) = \frac{1}{2\pi} \frac{d\phi}{dt}
\]

(2.48)

The modulation of the signal phase manifests itself in the sidebands of the oscillator carrier as offsets from the carrier frequency; these offsets are related to the multiples of angle-modulation frequency (Figure 19b).

A common but indirect representation of phase noise is denoted Power Spectral Density (PSD). This is the ratio of the noise power (in a 1 Hz bandwidth at an offset frequency from the carrier) to the total carrier power. This common
representation is applicable only to small phase deviations, where the relationship between the phase deviation and the sideband level is approximated as

\[
PSD \approx \frac{P_n}{P_c \text{Hz}} = \frac{\left\langle \cos (\omega t + \phi + \delta \phi) - \cos (\omega t + \phi) \right\rangle^2}{\left\langle \cos^2 (\omega t + \phi) \right\rangle} \approx \frac{\left\langle \sin^2 (\omega t + \phi) \delta\phi^2 \right\rangle}{\left\langle \cos^2 (\omega t + \phi) \right\rangle} = |\delta\phi|^2
\]

(2.49)

Figure 20: Phase noise represented in the time domain.

Usually oscillators show four distinct regions of phase noise (Figure 20):

- **Flicker FM noise** dominates the lowest frequency spectrum: the device’s flicker noise causes a random frequency modulation. This has a slope of \(1/f^3\).

- **White FM noise** is white noise that causes a random frequency modulation. This has a slope of \(1/f^2\).

- **Flicker PM noise** is modeled by flicker noise that mixes up to the oscillation frequency. This has a slope of \(1/f\).

- **White PM Noise**
• White PM noise is simply white noise that mixes up to the oscillation frequency. This has the typical flat white-noise floor.

We will consider the noise as $1/f^2$ since this is the effect of white noise applied to a nonlinear free running oscillator and what is experimentally observed in most cases.

The equations describing the noise behavior of a nonlinear VDP oscillator can be derived in two different ways. From one side the model can be modified to include an external current source to represent the internal noise generator. Kurokawa\textsuperscript{xxiv} showed that the resulting formulas have an in-phase term in the amplitude equation and a quadrature noise contribution in the phase dynamics.

\begin{equation}
\frac{dA}{dt} = \mu \omega_{3dB} A \left(1 - \frac{A^2}{\alpha^2}\right) + \omega_{3dB} \rho \cos(\Delta \phi) - \omega_{3dB} A G_n(t) = g(A, \phi) \tag{2.50}
\end{equation}

\begin{equation}
\frac{d\phi}{dt} = \omega_0 - \omega_{inj} + \omega_{3dB} \frac{\rho}{A} \sin(\Delta \phi) - \omega_{3dB} B_n(t) = f(A, \phi) \tag{2.51}
\end{equation}

where we defined $\omega_{3dB} \triangleq \omega_0/2Q$, half the 3-dB bandwidth of the oscillator tank circuit.

The same equations can be derived by adding a normalized ‘noise admittance’, $Y_n/G_L = G_n + jB_n$, into the oscillator embedding circuit (Figure 21), as described by Okabe and Okamura\textsuperscript{xxv}. 

54
Figure 21: Modified VDPO model to include the noise contributions.

From (2.50) and (2.51) it is clear that phase and amplitude noise are correlated, so AM, FM and PM noise will be present in an injected oscillator. However if the oscillator is free running, then the two noise contributions are uncoupled and the analysis will be easier.

The contribution to the noise from the injection source is also important:

Assuming that the amplitude and phase fluctuations due to these noise sources are small relatively to their steady state value, we can approach the problem in a similar way to the stability analysis, introducing a perturbation from the steady state situation, \( A = \hat{A} + \delta A \) and \( \phi = \hat{\phi} + \delta \phi \). The injection source will also have its own noise that can be represented as the perturbation around a nominal value, \( \rho = \hat{\rho} + \delta \rho \) and \( \psi = \hat{\psi} + \delta \psi \). To avoid worthless complications, we will assume by now a noiseless injection source. Recalling that in steady state only the noise is present, we can write
\[
\frac{d(\hat{A} + \delta A)}{dt} \approx \frac{d(\delta A)}{dt} \approx \frac{g(\hat{A}, \phi)}{\omega_{3dB} G_n(t)} + \frac{\partial g}{\partial \delta A}(\hat{A}, \phi) \delta A + \frac{\partial g}{\partial \delta \phi}(\hat{A}, \phi) \delta \phi
\]

\[
-\omega_{3dB} \hat{A} G_n(t) + \left[ \mu \omega_{3dB} \left( 1 - 3 \frac{\hat{A}^2}{\alpha^2} \right) + \omega_{3dB} G_n(t) \right] \delta A - \omega_{3dB} \rho \sin(\psi - \phi) \delta \phi
\]

\[
\frac{d(\hat{\phi} + \delta \phi)}{dt} = \frac{d(\delta \phi)}{dt} \approx \frac{f(\hat{A}, \phi)}{\omega_{3dB} B_n(t)} + \frac{\partial f}{\partial \delta A}(\hat{A}, \phi) \delta A + \frac{\partial f}{\partial \delta \phi}(\hat{A}, \phi) \delta \phi
\]

\[
-\omega_{3dB} B_n(t) - \omega_{3dB} \frac{\rho^2}{A^2} \sin(\psi - \phi) \delta A - \omega_{3dB} \frac{\rho^2}{A} \cos(\psi - \phi) \delta \phi
\]

Since \( \delta A \ll \hat{A} \), (2.52) can be further simplified. Fourier transforming and rearranging (2.52) and (2.53) we obtain

\[
\hat{\delta A} = -\frac{\hat{A} G_n + \rho \sin(\psi - \phi) \hat{\delta \phi}}{j \frac{\omega}{\omega_{3dB}} - \mu \left( 1 - 3 \frac{\hat{A}^2}{\alpha^2} \right)}
\]

\[
\hat{\delta \phi} = -\frac{\hat{B}_n + \frac{\rho^2}{A^2} \sin(\psi - \phi) \hat{\delta A}}{j \frac{\omega}{\omega_{3dB}} + \rho \frac{\rho}{A} \cos(\psi - \phi)}
\]

It’s important to notice that if \( \mu \) is small, the amplitude noise is amplified, since the system is prone to accept deviations from its steady state condition (see Figure 6): in this case the assumption \( \delta A \ll \hat{A} \) breaks down and the equations cannot be easily solved.

56
As last note, it is evident from (2.54) and (2.55) that the phase difference plays a main role in the overall noise performances: when $\Delta \phi = \phi - \hat{\phi} = 0$, then the equations are uncoupled, while if $|\Delta \phi| = 90^\circ$ the coupling is at his maximum.

Let’s now see what happens in the case of free running and injection locked to a low-noise reference source.

**Free-Running**

In the particular case of free running oscillator, the noise relationships are simplified

$$\delta A = - \frac{\alpha \tilde{G}_n}{j \frac{\omega}{\omega_{3dB}} + 2 \mu}$$

$$\delta \phi = - \frac{\tilde{B}_n}{j \frac{\omega}{\omega_{3dB}}}$$

The power spectral density (PSD) of the amplitude and phase fluctuation is found using the ensemble average of the modulus squared $\langle \delta \phi \cdot \delta \phi^* \rangle$ (see Appendix B4). Here for simplification we will omit $\langle \cdot \rangle$.

$$|\delta A|^2 = \frac{\alpha^2 |\tilde{G}_n|^2}{\left( \frac{\omega}{\omega_{3dB}} \right)^2 + 4 \mu^2}$$

(2.58)
\[
\left| \tilde{\phi} \right|^2 = \frac{\left| \tilde{B}_n \right|^2}{\left( \frac{\omega}{\omega_{3dB}} \right)^2}
\] (2.59)

Clearly the PM contribution dominates the noise close to the carrier, when \( \omega \ll \omega_{3dB} \). This explains why we are mostly interested in the phase noise performance of coupled oscillators.

For typical values of the parameters and assuming white noise of 1 W power equally distributed to 10 MHz, thus PSD of –70 dBc \( \left( |G_n|^2 = |B_n|^2 = 10^{-7} \text{ dBC/Hz} \right) \), we obtain the curves in Figure 22. It is clear that the AM noise is not important for typical values of the nonlinearity parameter \( \mu \), which usually is in the range 1-10. As we will see neglecting the amplitude noise lead to very good agreement with the measured data. Note that in F3dB and the 1/f noise intersects the white noise.
Figure 22: Noise floor (white noise), PM noise and AM noise for several values of $\mu$.

The phase noise can be simulated in Matlab with a small change in the model as shown in Figure 23.

A typical value for the phase noise of an oscillator is -60 dBC at 100 KHz from the carrier. In our case $F_{3dB} = 122.5$ MHz so the PM noise floor is 120 dBC/Hz as confirmed in our Matlab simulations (Figure 24).
Injection-Locked

When also the injection is taken into account, the analysis becomes complicated, but few assumptions can still be done for practical purposes.

As shown in Appendix B5, starting from (2.54) and (2.55), it is possible to find a condition to maintain relatively uncoupled AM and PM noise and make negligible the output amplitude random fluctuations compared to the phase noise. This condition if found to be $\rho \ll 2\mu$ and if this is the case, the PSD of the oscillator phase noise when the injection source is noiseless is given by

$$
|\tilde{\phi}|^2 = \frac{|\tilde{B}|^2}{\omega_2^2 + \frac{\rho^2}{A^2} \cos^2 (\psi - \hat{\phi})}
$$

(2.60)

Figure 24: Matlab simulation of the PM noise ($|\delta \phi|^2$) for a typical VDPO. Crossover occurs in $F_o$. 

Power Spectral Density [dB/Hz]

Offset [Hz] (from Carrier)

$F_c = 2.45$ GHz
$\alpha = 1$ V
$Q = 10$
$\mu = 6$
If now we consider the more realistic case of a low-noise injection source, then the equations become (Appendix B6)

\[
\left| \delta \phi \right|^2 = \left| \hat{B}_n \right|^2 + \frac{\rho^2}{A^2} \cos^2 \left( \hat{\psi} - \hat{\phi} \right) \left| \delta \bar{\psi} \right|^2 = \frac{\omega^2_{dB} + \frac{\rho^2}{A^2} \cos^2 \left( \hat{\psi} - \hat{\phi} \right)}{\omega^2_{dB} + \frac{\rho^2}{A^2} \cos^2 \left( \hat{\psi} - \hat{\phi} \right)}
\]  

(2.61)

A very interesting property arises from (2.61) confirmed by the results in . Close to the carrier, the oscillator phase noise tracks the behavior of the low noise injection source. Moving away from \( F_c \), the oscillator returns to its natural phase noise characteristics (free-running). This is a very important feature of injection-locked nonlinear oscillators and it will justify the results we will find for COA.
Figure 26: PM noise for a typical VDPO for several strengths of the injection low-noise source.

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The analysis of the microwave nonlinear oscillator will now be extended to an array of interacting elements. The analysis of multiple-oscillator systems involves relatively minor extensions of the elementary injection-locking theory, but yields significantly more complex and interesting dynamics. The discrete analysis of the system will be followed by the innovative continuous approach, which confirms the previous results in a more complete scenario. The important demonstrated features of COAs will be verified by Matlab implementation of the discrete and continuous models.

**Discrete Modeling of Coupled Oscillators Arrays**

As we previously stated, there are many different approaches to the theory of nonlinear oscillations but one is particularly well suited to microwave circuits. The analysis presented here strictly follows the theory developed by York and Liao who extended to coupled systems the method first utilized by Kurokawa to describe the operation and noise characteristics of microwave oscillators. The
frequency-domain impedance, admittance, or scattering parameter description of
the system is used to generate the amplitude and phase dynamic governing
equations as result of a rigorous expansion of the time-domain response through the
inverse Fourier transform.

**Coupling Network Theory**

The elements of COAs can be interconnected and/or injection locked to an
external source in a number of possible configurations. A few examples from the
literature are shown in Figure 27, such as a corporate fed externally-locked array\textsuperscript{iii},
unilateral nearest-neighbor injection-locking\textsuperscript{iv}, and “inter-injection-locked”\textsuperscript{v,vi} or
“mutually synchronized”\textsuperscript{vii,viii} bilateral nearest-neighbor injection-locking.
A general model for analysis is shown in. Each oscillator circuit is coupled to an $N$-port network, which will be described in terms of $Y$-parameters.
The \(N\)-port includes both coupling circuits and the load. At each port we define an oscillator admittance, \(Y_{osc,i}\), to describe the nonlinear active element and embedding circuit, which is coupled to a load admittance, \(Y_{L,i}\), the input admittance of the coupling network at the \(i^{th}\) port. The possibility of externally injected signals are included via the independent current sources \(I_{inj,i}\) at each port, which (if nonzero) are assumed to be coherent. From Kirchoff’s Current Law we can write

\[
I_{inj,i}(\omega) = V_{inj,i}(\omega) \left[ Y_{osc,i}(\omega, |V_i|) + Y_{L,i}(\omega) \right]
\]

Figure 28: (a) General model for oscillator array analysis, which includes the topologies of Figure 27 as special cases. The coupling network includes the load network (usually an antenna array). (b) A simple and practical parallel-coupling network using resistively-loaded transmission lines.
where the tilde (˜) denotes a frequency-domain (phasor) quantity. From linear network theory we can write

\[ Y_{L,i}(\omega) \sum_{j=1}^{N} Y_{ij}(\omega) \frac{\tilde{V}_j}{V_i} \]  

(3.2)

where \( N \) is the number of oscillators in the system. So (3.1) can be written as

\[ \tilde{I}_{\text{inj},i}(\omega) = \sum_{j=1}^{N} Y_{t,ij}(\omega,|V_i|) \hat{V}_j(\omega) \]  

(3.3)

where

\[ Y_{t,ij}(\omega,|V_i|) = Y_{\text{osc},ij}(\omega,|V_i|) \delta_{ij} + Y_{ij}(\omega) \]  

(3.4)

and \( \delta_{ij} \) is the Kronecker delta. The next critical assumption in the analysis is that each oscillator is a single-mode system, designed to produce nearly sinusoidal oscillations around a nominal center frequency \( \omega_i \) his is the “free-running” or unperturbed oscillation frequency of the \( i \)th oscillator. Most practical oscillators can be designed to satisfy these criteria by using an embedding circuit with a well-defined and isolated resonance at \( \omega_i \) with a sufficiently high \( Q \)-factor (\( Q > 10 \)). The device must be terminated to provide narrowband gain around this resonance. The time-dependent output voltage can then be written (using complex notation) in the following useful forms

\[ V_i(t) = A_i(t) e^{j[\omega_i t + \phi_i(t) - \theta_i]} = A_i(t) e^{j\theta_i(t)} = V_i'(t) e^{j\theta_i t} \]  

(3.5)
where $A_i$ and $f_i$ are dynamic amplitude and phase variables, $V_i'(t) = A_i(t)e^{j\phi_i(t)}$ is the output “phasor” voltage at port $i$, $\theta_i(t) = \omega_r t + \phi_i(t)$ is the instantaneous phase, and $\omega_r$ is a “reference” frequency that is presumably close in magnitude to the average $\omega_i$, but otherwise somewhat arbitrary and chosen for convenience for a particular problem. The reason for defining $V_i(t)$ in terms of this reference frequency will become clear later, but in the meantime can be considered as establishing a variable harmonic reference for the time variation of $f$, which proves convenient when the oscillator system is perturbed from its free-running state. The true time variation is obtained by taking the real part of (3.5).

Applying the inverse Fourier transform to (3.3) and exploiting the slowly varying amplitude and phase assumption, it can be shown (see Appendix A) that (3.3) transforms

$$(3.6)$$

where we have used the notation in (3.5). As described in the Appendix A, (3.6) is an approximation. Higher order time derivatives of amplitude and phase, and higher order frequency derivatives of the total admittance, must be negligible compared with the first two terms of the expansion (the two terms in the square brackets in (3.6). This is generally satisfied for oscillation around an isolated resonance and for amplitude and phase fluctuations slow compared to the carrier (an excellent assumption in practice). This result is equivalent to using a
“Kurokawa substitution” for the frequency in (3.3) and can be written in the matrix notation

\[
\mathbf{I}_{\text{inj}}(t) = \sum_{j=1}^{N} Y_t(\omega_r, A_j) \cdot \mathbf{V}' + \frac{dY_t}{d(j\omega)} \cdot \frac{d\mathbf{V}'}{dt} e^{j\omega t}
\]  

(3.7)

where \( Y_t \) is a matrix with elements \( Y_{t,ij} \), \( \mathbf{I}_{\text{inj}}(t) \) is a vector with elements \( Y_{t,ij}(t) \), and \( \mathbf{V}' \) is a vector with elements \( V'_j \) (the phasor port voltages). This is a coupled set of first-order differential equations for the port voltages, and can be written as

\[
\frac{d\mathbf{V}'}{dt} = \left( \frac{\partial Y_t}{\partial (j\omega)} \right)^{-1} \left[ \mathbf{I}_{\text{inj}} e^{-j\omega t} - \mathbf{Y}_t \cdot \mathbf{V}' \right]
\]

(3.8)

This in turn can be expressed as

\[
\frac{d\mathbf{V}'}{dt} = F_i(\mathbf{V}) \mathbf{V}' \quad i = 1...N
\]

(3.9)

where

\[
F_i(\mathbf{V}) = \sum_{j=1}^{N} \left( \frac{\partial Y_t}{\partial (j\omega)} \right)^{-1} \left[ \frac{\mathbf{I}_{\text{inj,j}}}{\mathbf{V}'_i} - \sum_{k=1}^{N} Y_{t,k} \frac{V'_k}{V'_i} \right]
\]

(3.10)

This can be solved for the amplitude and phase variations by separating real and imaginary parts to give

\[
\frac{dA_i}{dt} = A_i \Re F_i(\overline{A}, \overline{\phi})
\]

(3.11)
This is a general result for analysis of amplitude and phase dynamics in coupled oscillator systems. We can simplify the analysis further by restricting attention to so-called "broadband" coupling networks, which satisfy

\[
\frac{\partial Y_{t,ij}}{\partial (j\omega)} \approx \frac{\partial Y_{osc,i}}{\partial (j\omega)} \delta_{ij}
\] (3.13)

In other words, such coupling networks have negligible frequency dependence near the operating frequency. This is equivalent to saying that the oscillator \(Q\)-factor is the dominant \(Q\) of the system (see Lynch\textsuperscript{x} for a treatment of coupling networks with a non-negligible frequency dependence, such as resonant coupling through an external cavity). With this assumption, \(F_i\) in (3.10) is given

\[
F_i(\vec{V}) \cong \left( \frac{\partial Y_{osc,i}(\omega_r, A_i)}{\partial (j\omega)} \right)^{-1} \left[ \sum_{j=1}^{N} Y_{ij}(\omega_r) \frac{V_j}{V_i} \right] (3.14)
\]

We are most interested in steady-state solutions to (2.10)-(2.11) where all oscillators are synchronized to a common frequency, \(\omega\), which occurs when

\[
\frac{dA_i}{dt} = 0 \quad \& \quad \frac{d\phi_i}{dt} = \omega - \omega_r \quad i = 1\ldots N
\] (3.15)

In this case the oscillator phases will be bounded in time if the reference frequency is chosen to be \(\omega_r = \omega\), in which case the steady-state solutions are determined by the set of nonlinear algebraic
If externally injected signals at frequency $\omega_{\text{inj}}$ are present, the synchronized frequency (and reference frequency) can be taken as $\omega = \omega_{\text{inj}} = \omega_r$. However, in the absence of externally injected signals ($I_{\text{inj},i} = 0$ for all $i$), then the steady-state synchronized frequency is not known a priori, but must be determined from (3.15). The real and imaginary parts of $F_i$ must separately equate to zero, so (3.15) represents a set of $2N$ equations. Since the amplitude and phase of each oscillator are $2N$ unknowns that must be solved for, it would appear that when $\omega$ is also unknown there would be $2N+1$ unknowns. In this case, however, one of the phases is arbitrary and can be set to zero (only the relative phases are important physically). For externally locked arrays, all of the oscillator phases are unknown, since the injected signals establish a phase reference.

Unless the oscillators are strongly coupled, the oscillation amplitudes will remain close to the free-running values. Assuming this the case, the amplitude and phase dynamics are uncoupled, to first order. We then can restrict attention to the phase dynamics governed by (3.12). This will simplify the analysis considerably.

**Single Cell Modeling**

To proceed further, we need to specify an oscillator model. In a well-designed oscillator, the oscillation frequency will occur in the vicinity of a reactance or susceptance null, which is assumed to be well isolated from other nulls (spectrally)
to avoid mode-hopping or multi-frequency operation. In a narrow range of frequencies around such a resonance, by either a parallel or a series resonant circuit, shown in Figure 29a,b, respectively can model the oscillator. As we saw in the previous chapter, for stability in the free-running case, the device in the parallel model must have a negative conductance that decreases with increasing oscillation amplitude. The device in the series model must have a negative resistance that decreases in magnitude with increasing oscillation amplitude. We assume that the array is composed of such oscillators, which are stable in their free-running state. The coupling between the oscillators serves only to synchronize the frequencies via the injection-locking phenomenon, and results in only a slight perturbation of the oscillators from the free-running configuration.

Figure 29: Parallel (a) and series (b) equivalent models for the single cell in COAs.
Parallel Oscillator Model

Using the parallel model of Figure 29a, the input admittance of the ith oscillator near the resonant frequency $\omega_i$ can be approximated as

$$Y_{osc,i} \approx -G_d(A_i) + j2C_i(\omega - \omega_i) \quad (3.17)$$

where $C_i$ is the shunt capacitance. In the free-running state, the oscillator feeds a load conductance of $G_L$ as shown, so we can define a $Q$-factor for the free-running oscillator as $Q = \omega_i R_d C_i$ and write

$$Y_{osc,i} \approx -G_d(A_i) + j\frac{G_L}{\omega_{3dB}}(\omega - \omega_i) \quad (3.18)$$

where $\omega_{3dB} = \omega_i / 2Q$ is half the 3 dB bandwidth of the oscillator tank circuit.

We assume that the oscillators can have slightly different free-running frequencies, but that the $Q$-factors and 3 dB bandwidths are all the same, to first order. From this, we see that

$$\frac{\partial Y_{osc,i}(\omega_r)}{\partial (j\omega)} = \frac{G_L}{\omega_{3dB}} \quad (3.19)$$

Using (3.18), (3.19) and (3.2), the function $F_i$ from (2.13) is then given by

$$F_i(V) \approx j(\omega_i - \omega_r) + \frac{\omega_{3dB}}{G_L} \left[ \frac{I_{inj,i}}{V_i'} + G_d(A_i) - \sum_{j=1}^{N} \frac{A_j}{A_i} e^{j(\theta_j - \theta_i)} \right] \quad (3.20)$$
**Series Oscillator Model**

The dynamics for a series oscillator model can be derived in similar fashion. Close to the resonant frequency ($\omega = \omega_r$) the input impedance of the oscillator in Figure 29b is given by

$$Z_{osc,i} \approx -R_d(A_i) + j \frac{R_i}{\omega_{3dB}}(\omega - \omega_r)$$  \hspace{1cm} (3.21)

Our formulation requires the input admittance and derivative with frequency. Assuming that the oscillators are only slightly perturbed from their free-running state, the amplitude of oscillation will remain close to the free-running value, and therefore $-R_d(A_i) \approx R_L$. Using this approximation, the admittance can be expressed as

$$Y_{osc,i} = \frac{1}{Z_{osc,i}} \approx -\frac{1}{R_d(A_i)} = j \frac{\omega - \omega_i}{\omega_{3dB}R_L}$$  \hspace{1cm} (3.22)

and so

$$\frac{\partial Y_{osc,i}(\omega)}{\partial (j\omega)} = -\frac{1}{\omega_{3dB}R_L}$$  \hspace{1cm} (3.23)

Combining (3.23), (3.2) and (3.14) gives

$$F_i = j(\omega_i - \omega_r) \frac{\omega_{3dB}}{G_L} \left[ I_{inj,i} \left( \frac{1}{V_i'} + \frac{1}{R_d(A_i)} \right) + \sum_{j=1}^{N} \frac{Y_{ij} A_j'}{A_i}' e^{j(\theta_j - \theta_i)} \right]$$  \hspace{1cm} (3.24)
The only difference between (3.24) and (3.20) is the sign proceeding the bracketed term. Nevertheless, this has an important influence on the phase relationships, as we will see.

**Broadband Nearest-Neighbor Coupling**

Oscillators can be interconnected in a broadband-coupling network in several ways. York et al.\(^x\) developed the analysis for general coupling networks. We will restrict attention to a simple nearest-neighbor coupling network that is appropriate for planar oscillator circuits, shown in Figure 28b. When the coupling resistance is chosen such that \(R = Z_0\), this network has the following admittance parameters\(^{xi}\)

\[
Y_{ij} = \begin{cases} 
\frac{\eta_i + G_L}{2Z_0} & i = j \\
\frac{-e^{-j\beta L}}{2Z_0} |i - j| = 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
\frac{\partial Y_{ij}}{\partial (j\omega)} = \begin{cases} 
0 & i = j \\
\frac{\tau_g e^{-j\beta L}}{2Z_0} & |i - j| = 1 \\
0 & \text{otherwise}
\end{cases}
\]  

(3.25)

where \(\beta L\) is the electrical length of the transmission-line, \(Z_0\) is the characteristic impedance, \(\eta_i = 2 - \delta_{i1} - \delta_{iN}\), \(\delta_j\) is the Kronecker delta function, and \(\tau_g\) is the group delay through the transmission-line. For TEM or quasi-TEM lines, \(\tau_g = \beta L/\omega\), and when \(\beta L \leq 2\pi\) then the assumptions leading to (3.14) are satisfied if \(\pi R_L/QZ_0 \ll 1\) \((R_L = 1/G_L)\), where \(Q\) is the \(Q\)-factor of the oscillator. We will
assume this constraint holds, and define the following “coupling parameters” for future convenience

\[ \epsilon \equiv \frac{R_L}{2Z_0} \text{ and } \Phi \equiv \beta L \] (3.26)

The independent sources representing externally injected signals are assumed mutually coherent at a common frequency \( \omega_{inj} \), and are written as

\[ I_{inj,i} = \rho_i G_L e^{j[\omega_{inj}t + \psi_i(t)]} \] (3.27)

Substituting (3.25) and (3.27) into (3.20) and (3.24), and using (3.26) the phase dynamics in (3.12) are described by

\[ \frac{d\phi_i}{dt} = (\omega_i - \omega_r) \mp \epsilon \omega_{3dB} \sum_{j=i-1}^{i+1} \sin \left( \Phi + \phi_i - \phi_j \right) \mp \rho_i \omega_{3dB} \sin \left( \phi_i - \psi_j \right) \quad i = 1...N \] (3.28)

where we have assumed uniform amplitudes \( A_i = A_j \). The upper sign is for parallel oscillators (Figure 30a), while the lower sign is for series oscillators (Figure 30b). This equation applies to the end elements in the array if any terms containing subscripts 0 or \( N+1 \) are ignored.

As for the single element of the previous chapter, the arbitrary choice of the reference frequency allows for analytical simplifications. It is understood that \( \omega = \omega_{inj} \) when externally injected signals are present. On the other hand, with no injection signals present (\( \rho_i = 0 \) for all \( i \)), the reference should be selected as the
average of the free-running frequencies, since adding in steady state all $N$ equations in (3.28) yields the interesting result

$$\omega = \frac{1}{N} \sum_{i=1}^{N} \omega_i$$  \hspace{1cm} (3.29)

which is the synchronization frequency. In the transient $\omega$ is an unknown to be solved for otherwise.

Figure 30: Nearest-neighbor coupled-oscillator systems considered in this chapter. (a) Parallel-oscillators coupled in parallel using the resistively loaded network of Figure 28b; (b) Series-oscillators in a similar coupling network.
York and Liao demonstrated theoretically and experimentally that desired phase progressions could be established in free-running \((\rho_i = 0)\) nearest neighbor coupled arrays. It can be achieved using coupling networks like that of Figure 28b, by properly detuning the oscillator free-running frequencies prior to synchronization. The coupling phase \(\Phi\) also plays an important role in determining the stable range of phase shifts and the frequency distribution required for implementation. For example, using (3.28) for parallel oscillators we showed that beam-scanning around the broadside direction can be achieved when \(\Phi = 0^\circ\) with only detuning of peripheral array elements, with the range of stable inter-element phase shifts being \(90^\circ < \Delta \phi < 90^\circ\). Note that the phase dynamics (3.28) for series-type oscillators with \(\Phi = 180^\circ\) would yield exactly the same phase dynamics as the parallel-type with \(\Phi = 0^\circ\) (and vice versa). This means that series oscillators require a significantly different coupling circuit than parallel oscillators in order to produce the same phase distributions.

For illustrative purposes as well as practical merit, we will focus on the case where \(\Phi = n\pi\), where \(n\) is an integer (the following analysis will be accurate for small deviations in coupling phase around the point \(\Phi = n\pi\)). In this case, (3.28) becomes

\[
\frac{d\phi_i}{dt} = (\omega_i - \omega_r) + \epsilon \omega_{3dB} \cos \Phi \sum_{j=1}^{i-1} \sin (\phi_i - \phi_j) + \rho_{ij} \omega_{3dB} \sin (\phi_i - \psi_i) \quad i = 1...N
\]

(3.30)
The phase relationships will be strongly dependent on the free-running frequencies, $\omega_i$. In a practical array the free-running frequency distribution can be dynamically manipulated by employing voltage-controlled oscillators (VCOs) with a varactor as tuning element. For a given coupling network, we have the following analysis and synthesis problems:

- given a set of free-running frequencies, solve for the steady-state phase;
- given a desired phase distribution, solve for the free-running frequencies that will produce this phase relationship.

The former is extremely difficult in the general case, and must be carried out numerically (an alternative based on continuum modeling is discussed later). The linearity of (3.30) with respect to the free-running frequencies makes the synthesis procedure very straightforward.

For steady-state synchronous oscillation at $\omega$, substituting a desired phase relationship, denoted by a circumflex $\hat{\phi}_i$, into (3.30) gives the required free-running frequencies

$$\omega_i = \omega \pm \varepsilon \omega_{3dB} \cos \Phi \sum_{j=i-1}^{i+1} \sin \left( \hat{\phi}_i - \hat{\phi}_j \right) \pm \rho \omega_{3dB} \sin \left( \phi_i - \psi_i \right) \quad i = 1...N \quad (3.31)$$

It would seem to imply that we could synthesize any arbitrary phase relationship by suitably adjusting the free-running frequencies, but that is not the case; each possible solution must be checked for stability.
Note that for a single oscillator ($\varepsilon = 0, N = 1$), both series and parallel models yield Adler's equation described in the first chapter

$$\frac{d\phi}{dt} = (\omega_0 - \omega_{inj}) \mp \rho \omega_{2dB} \sin(\phi - \psi) \quad (3.32)$$

The difference in sign in this case is attributed to the assumed direction of the injected current (into the oscillator). That is, the sign depends on how we define the phase of the injected signal. This is also true for arrays; we can always uniformly shift the injected-signal phases by $\pm \pi$ to change the sign of the injection terms in (3.30). This is another way of saying that the phase reference in the injected system is arbitrary, and therefore the sign of the injection terms is not considered especially important.

**Continuum Modeling of Coupled Oscillators Arrays**

The previous discrete description of COAs behavior is sufficiently complicated that numerical simulations are needed to obtain a qualitative idea of their dynamics. Moreover the full nonlinear formulation makes the intuitive understanding of the phenomena exceedingly difficult.

It has recently been pointed out that the analysis can be greatly simplified if one proceeds to the so-called continuum limit in which the array phase is represented by a continuous function\textsuperscript{xii}. York and Pogorzelski noted that (3.30) could be expressed in matrix form using an operator resembling a discretized version of the
familiar Laplacian operator and conjectured that, as a consequence, solutions of Laplace’s equation may play a significant role in the behavior of the array. As a consequence of this conjecture, in steady state, a correct approximate description emerges as Poisson’s equation in which the distribution of the free-running frequencies of the oscillators appears as a source term analogous to charge density in electrostatics.

Rand et al.\textsuperscript{xiii} have described a related approach, where a discrete array is modeled as a continuum of oscillators governed by a single global differential equation, in a mathematical description of certain biological systems. Interestingly, their equations are virtually identical in form to those described by York and Pogorzelski.

Reducing the problem to that of solving Poisson’s equation is remarkably useful.

First, considerable insight into the operation of these arrays can be obtained by analogy to the corresponding electrostatic problem. For example, we will see that previously reported beam-scanning techniques, which are difficult to explain intuitively using the discrete modeling, are reduced to an equivalent parallel-plate capacitor problem.

In addition, since the system is described by a single differential equation, a new spectrum of analytical tools can be brought to bear on the problem, and this
allows us to quantify both the steady state and dynamic behaviors of the array in
new ways that increase our understanding of coupled-oscillator systems.

Lastly, although not treated here, the continuum analysis has been lately
generalized to two-dimensional arrays\textsuperscript{xiv,xv,xvi}, resulting in a considerable
computational advantage for large 2D array systems.

In this section we reveal the formulation of the continuum limit description of
coupled oscillator arrays. Both the mutual coupling as the external injection will be
included in our formulation. A quick look to the expected behavior of COAs under
single element detuning and injection is presented under the assumption of infinite
array length. Green’s functions and Laplace transform techniques will be used in
the to solve for the transient analysis. We will show that the steady state solution
could be found also with a simplified treatment directly from the Poisson’s
equations.

In the next sections this method, applied to the more realistic case of finite
length array, will be leading to analytic solutions for the phase evolution in both the
free-running and injection locked array. The results will be presented along with
the ones derived from the discrete modeling. The steady state will be included as a
particular case of the general solution for $t \to \infty$.

Here we will strictly follows the treatment of the continuum model presented in
\textsuperscript{xvii,xviii,xix,xx}. The mathematical details associated with the determination of the close
form solutions can be found in the original papers.
Derivation of Governing Equation

In the previous section we described in detail the linear array illustrated schematically in Figure 31 using a coupled set of nonlinear differential equations. These equations are derived by first describing the behavior of an individual oscillator with injection locking in the manner of Adler\textsuperscript{xxi} and then allowing the injection signals to be provided by the neighboring oscillators in the array. For sake of symmetry and thus easier interpretation of the results, the array extends in the ranges \(-N \leq i \leq N\) in the discrete case and \(-a-1/2 \leq x \leq a+1/2\) for continuum model: The respective lengths will be \(2N+1\) and \(2a+1\). The array is taken to extend over unit cells with an oscillator at the center of each unit cell. This leads to the range of \(x\) noted above.

![Diagram of coupled oscillator array with the \(p\)th oscillator locked to an externally derived signal.]

The important assumption here is that the amplitude equations do not play a main role in the array overall dynamics. As we know from the previous chapter, this is true if mutual coupling and injection strength are relatively weak. With this assumption, rewriting the phase equations, encapsulated in (3.28), for the general case results in the coupled set
\[
\frac{d\phi_i}{dt} = (\omega_i - \omega_r) + \sum_{j \neq i}^{i+1} \frac{\alpha_j}{\alpha_i} e_{ij} \omega_{3dB,i} \sin (\Phi_{ij} + \phi_i - \phi_j) + \frac{\rho_i}{\alpha_i} \omega_{3dB} \sin (\phi_i - \psi_i) \quad i = 1 \ldots N
\]  

(3.33)

Defining the coupling (inter-oscillator) and the injection locking range as

\[
\Delta \omega_{lock,ij} = \frac{\alpha_j}{\alpha_i} e_{ij} \omega_{3dB,i} = \frac{\alpha_j}{\alpha_i} \frac{e_{ij} \omega_i}{2Q}
\]

\[
\Delta \omega_{lock,i,inj} = \frac{\rho_i}{\alpha_i} \omega_{3dB,i} = \frac{\rho_i}{\alpha_i} \frac{e_{ii} \omega_i}{2Q}
\]

and assuming parallel model for the single cell (the results would be the analogous for the series model), (3.33) can be rearranged to obtain

\[
\frac{d\phi_i}{dt} = (\omega_i - \omega_r) - \sum_{j \neq i}^{i+1} \Delta \omega_{lock,ij} \sin (\Phi_{ij} + \phi_i - \phi_j) - \Delta \omega_{lock,i,inj} \sin (\phi_i - \psi_i) \quad i = 1 \ldots N
\]

(3.35)

Obviously if only few sources are injected in positions denoted with \(p\), then the use of the discrete Kroneker delta function \(\delta_{ip}\) will provide a simplified version of in each of the cases treated below. Each of the injection contributions will then be expressed as \(-\delta_{ip} \Delta \omega_{lock,p,inj} \sin (\phi_p - \psi_p)\).
A further simplification arises from the realistic and practical case of zero coupling phase, $\Phi = \Phi_{ij} = n\pi$, and identical coupling locking ranges $\Delta \omega_{lock} = \Delta \omega_{lock,ij}$.

With one externally derived injection signal we obtain

$$\frac{d\phi_i}{dt} = (\omega_i - \omega_r) - \Delta \omega_{lock} \sum_{\substack{j=i-1 \atop j\neq i}}^{i+1} \sin(\phi_i - \phi_j) - \delta_{ip} \Delta \omega_{lock, p, inj} \sin(\phi_p - \psi_p) \quad i = 1...N$$

(3.36)

The most important assumption in the continuum modeling is that the inter-oscillator phase differences are small in order to approximate the sine function by its argument. In reality between two oscillators the difference can be up to $\pi/2$, but here we suppose that there is an infinite number of oscillators, infinitely close to each other. Real oscillators will be then positioned with a finite distance, thus the phase difference can be anything.

Using this idea, $\sin(\phi_i - \phi_j) \approx \phi_i - \phi_j$ and (3.36) can be rewritten in the form

$$\frac{d\phi_i}{dt} = (\omega_i - \omega_r) + \Delta \omega_{lock} (\phi_{i+1} - 2\phi_i + \phi_{i-1}) - \delta_{ip} \Delta \omega_{lock, p, inj} \sin(\phi_p - \psi_p) \quad i = 1...N$$

(3.37)

At this point, we note that the quantity in parentheses is merely a finite-difference approximation for the second derivative of the phase with respect to a spatial variable $x$, which corresponds to the index at integer values. Thus, (3.37)
can now be easily recognized as the finite-difference approximation corresponding to the partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial \phi}{\partial \tau} = -\frac{\omega_{\text{tune}} - \omega_r}{\Delta \omega_{\text{lock}}} + \delta_{p} \frac{\Delta \omega_{\text{lock,p,inj}}}{\Delta \omega_{\text{lock}}} \left( \phi - \psi_p \right)$$  \hspace{1cm} (3.38)

where we normalized (3.37) to the locking range, defining the unitless time \( \tau = \Delta \omega_{\text{lock}} t \) and we defined the tuning distribution function \( \omega_{\text{tune}}(x, \tau) \).

Now, let the injection signal be represented by \( V(x) \psi(\tau) \), where the time dependence of the injection signal phase is given by \( \psi \) and \( V \) gives the spatial distribution of the injection signals. For analytical convenience, the Dirac delta is used instead of a pulse one-unit cell wide. Thus the injection terms in (3.38) become

$$V(x) = \frac{\Delta \omega_{\text{lock,p,inj}}}{\Delta \omega_{\text{lock}}} \delta(x - p)$$  \hspace{1cm} (3.39)

And (3.38) reduces to

$$\frac{\partial^2 \phi}{\partial x^2} - V(x) \phi - \frac{\partial \phi}{\partial \tau} = -\frac{\omega_{\text{tune}} - \omega_r}{\Delta \omega_{\text{lock}}} - V(x) \psi(\tau)$$  \hspace{1cm} (3.40)

Note that \( \phi(x, \tau) \) is the phase across the array as a function of the normalized time, while the driving functions are the distribution of the oscillator free-running (tuning) frequencies relative to the reference frequency and the injection signals.
The boundary conditions at the ends of the array must be specified to solve (3.40) and are most easily derived via the following artifice.

Imagine one additional fictitious oscillator added at each end of the array. Let these additional oscillators each be tuned as a function of time in such a manner as to maintain the oscillator phase equal to the phase of the corresponding actual end oscillator. The resulting injection signals from the fictitious oscillators will then have no effect on the dynamics of the actual array. In this sense, the extended array simulates the actual array. However, since the phases of the fictitious oscillator at each end and the corresponding end oscillator are maintained equal, the phase gradient at the ends of the array is seen to be zero. Therefore, the boundary conditions at the array ends $x = -a - 1/2$ and $x = a + 1/2$ are seen to be the classical Neumann conditions independent of time.

Finally, we emphasize that the continuous function only has meaning at integer values of where it takes on the value of the phase of the oscillator of index $n = x$. The above development was carried out for an odd number of oscillators. If the number of oscillators is even, the theory as developed can be applied by setting $a = M/2$ where $M$ is the number of oscillators. In that case, however, the solution only has meaning when $x$ is a half-integer where it takes on the value of the phase of the oscillator indexed by that value of $x$. 

88
Synchronization and Steady State Solution

When no injection is present, averaging (3.40) over the length of the array results

\[
\left(\frac{\partial^2 \phi}{\partial x^2}\right) - \frac{\partial \langle \phi \rangle}{\partial \tau} = -\frac{\langle \omega_{\text{tune}} \rangle - \omega_r}{\Delta \omega_{\text{lock}}} \tag{3.41}
\]

The first term is zero because the integral of the second derivative of the phase over the length of the array is equal to the difference of the phase gradients evaluated at the endpoints, and they are both zero by virtue of the Neumann boundary condition there.

Now, by definition, the normalized instantaneous frequency of the oscillators is given by

\[
\frac{\omega}{\Delta \omega_{\text{lock}}} = \frac{d \phi}{d \tau} + \frac{\omega_r}{\Delta \omega_{\text{lock}}} \tag{3.42}
\]

Substituting (3.42) into (3.41), we have that

\[
\omega = \langle \omega_{\text{tune}} \rangle = \frac{1}{2a+1} \int_{-a-1/2}^{a+1/2} \omega_{\text{tune}}(x) \, dx \tag{3.43}
\]

In steady state all of these mutually locked oscillators will oscillate at the same frequency: this implies that the ensemble frequency is merely the average of the tuning frequencies, a result that has been previously obtained using the discrete model in (3.29). Since we do not want time variation in steady state, it will be convenient to set the reference frequency equal to the initial value of this ensemble
frequency when the oscillator is free running. When injection is present then 
\[ \omega = \omega_{\text{inj}}. \]

The steady-state solution obtained as a limit of the dynamic solution can also be obtained directly by solution of Poisson’s equation. Therefore, the differential equation (3.40) takes the particularly simple form

\[ \frac{d^2 \phi}{dx^2} = -\rho \]  \hspace{1cm} (3.44)

where we defined \( \rho = \Omega_{\text{tune}} - \Omega_0, \quad \Omega_{\text{tune}}/\Delta \omega_{\text{lock}}, \) and \( \Omega_0 = \langle \Omega_{\text{tune}} \rangle, \) i.e., Poisson’s equation of electrostatics where \( \phi \) is the analog of electrostatic potential and \( \rho \) is the analog of charge density (divided by permittivity). This charge density is determined by the deviations of the tuning frequencies from \( \Omega_0 \) and, as such, its integral over the array is clearly zero, implying zero net charge

\[ \int_{-a-1/2}^{a+1/2} \rho(x) \, dx = 0 \]  \hspace{1cm} (3.45)

It is noted at this point that one may determine the tuning necessary to obtain any desired phase distribution along the array by merely substituting the desired phase function into (3.44) and differentiating twice, with respect to \( x \), to obtain the tuning function.
Following this line of thinking, we can argue that as long as the exterior region $|x| > a + 1/2$ is charge neutral such that $\rho(x) = 0$, then there must be zero electric field in the exterior region, in other words,

$$\frac{d\phi}{dx} = 0 \quad \text{for} \quad |x| > a + 1/2 \quad (3.46)$$

In the language of injection locking, this suggests that we can imagine a finite coupled oscillator chain as an infinite chain where all the oscillators in the exterior region have a free-running frequency equal given by (3.43). This seems reasonable, since we know from simple injection-locking theory that injecting a coherent signal into an oscillator (that is, at exactly the same frequency) does not perturb the oscillator. We can also show that this works for an array by direct substitution in the discrete model. In the electrostatic analogy, this corresponds to placing a perfect conductor at the boundaries.

We will see in the next section that it will be of particular interest, to observe the phase distributions obtainable by tuning only two of the oscillators—those at the two ends of the array and these electrostatic analogy will be used to confirm the solution obtained as the limit of the complete solution for the free-running array.

**A First Glimpse to the Dynamics**

An idea of the transient response of the array to detuning and injection can be readily obtained with the assumption that the array is infinitely long. Later this
assumption will be removed to obtain more accurate study of realistic situations in order to compare the results with the discrete analysis.

**Single Detuning of Free-running Infinite Array**

Consider an array for which \( a = \infty \), i.e., a linear array of infinite length. Assuming no injection is present, let the oscillator at \( x = b \) be detuned by an amount \( C \) (measured in locking ranges) from the ensemble frequency at \( t = 0 \). This implies that the driving function for (3.40) may be represented by

\[
- \frac{\omega_{\text{tun}} - \omega_{r}}{\Delta \omega_{\text{lock}}} = -C u(\tau) \delta(x - b)
\]  

(3.47)

Note that while it is not limited in this linearized theory, \( C \) is, in reality, limited by the fact that the magnitude of the sine function in (3.36) must be less than or equal to unity.
Transforming in the Laplace domain, solving for the transform of $\phi$ using the Green’s function and then returning in the time domain we obtain

$$
\phi(x, \tau) = C \left\{ 2 \sqrt{\frac{\tau}{\pi}} e^{-\frac{(x-b)^2}{4\tau}} - |x-b| \text{erfc} \left( \frac{|x-b|}{2\sqrt{\tau}} \right) \right\} u(\tau) \tag{3.48}
$$

The behavior of this function over the ranges $0 \leq \tau \leq 250$ and $-10 \leq x \leq 10$ is shown in Figure 32a. Note that at infinite time, this function diverges as the square root of the time. That is, the phase never reaches a steady-state value. However,
differentiating this function with respect to time and recalling (3.42) give the
dynamic behavior of the frequency in the form

\[ \omega(x, \tau) = \omega_r + \Delta \omega_{lock} \frac{C}{\sqrt{\pi \tau}} e^{-\frac{(x-b)^2}{4\tau}} u(\tau) \]  

(3.49)

which at infinite time converges to the steady-state value equal to the original
ensemble frequency as one over the square root of the time. This is a manifestation
of the fact that changing the tuning of one oscillator in an infinite array does not
change the ensemble frequency. This frequency distribution is shown in Figure
32b. At this point, one might question the use of the Dirac delta function to
represent the spatial distribution of the detuning in (3.47), preferring instead the use
of a unit amplitude square pulse one unit cell wide. The result of using this
alternate representation can be readily obtained from (3.48) by numerical
convolution and may thus be seen to differ very little from (3.48) itself. In the
mentioned papers it is shown that the difference is not significant and thus in this
treatment to use the delta function representation for analytical convenience.

**External Injection of Infinite Array**

Let now all of the oscillators be tuned to the same frequency \( \omega_0 \). Now, inject
into the oscillator at \( x = b \) an externally derived signal of frequency \( \omega_0 \Delta \omega_{lock} u(\tau) \).
Thus, prior to \( \tau = 0 \), all the oscillators will be in phase. Subsequently, based on
(3.40), the behavior of the oscillators will be given by the solution to the partial
differential equation
\[
\frac{\partial^2 \phi}{\partial x^2} = C \delta(x-b) \phi - \frac{\partial \phi}{\partial \tau} = -C_0 C \delta(x-b) u(\tau)
\]  
(3.50)

where we chose \( \omega_r = \omega_0 \). For later convenience, we now define the normalized frequency divergence from the injected frequency as

\[
\tilde{\omega}(x, \tau) = \omega(x, \tau) - C_0 u(\tau) = \frac{\partial \phi}{\partial \tau} - C_0 u(\tau)
\]  
(3.51)

Solving using again the Laplace domain method, we find solution as sum of homogeneous and particular contributions

\[
\tilde{\omega}(x, \tau) = C_0 \left[ \text{erfc} \left( \frac{|x-b|}{2\sqrt{\tau}} \right) - e^{(|x-b|^2/2)} e^{(C^2/4)} \text{erfc} \left( \frac{C \sqrt{\tau}}{2} + \frac{|x-b|}{2\sqrt{\tau}} \right) - 1 \right]  
\]  
(3.52)

For each fixed value of \( x \), this function begins at a value of \(-C_0\) when \( \tau = 0 \) and evolves smoothly and monotonically toward a final value of zero at infinite \( \tau \). This transition from the reference frequency to the injection frequency occurs first for values near zero and later for oscillators more distant from the center of the array. This is, of course, to be expected since the effect of the sudden switch in injection signal frequency would be expected to diffuse from the injection point outward in both directions along the array. The diffusion rate is governed by the ratio of \( t \) to \( \tau \), i.e., the interoscillator locking range \( \Delta \omega_{lock} \). The phase behavior of the array can be obtained as the time integral of this function. Interestingly, however, the resulting function approaches infinity for \( \tau \to \infty \), indicating that the phase never reaches a steady-state value, as does the frequency. Rather, it continues
to evolve for all time as it did for the detuning step. Specifically, for late times, the frequency differs from the injection frequency as one over the square root of the time, which implies that the phase, which is its time integral, differs from the injection phase as the square root of time, which, of course, approaches infinity for infinite time.

Figure 33: Phase (a) and frequency divergence (b) response to injection of an infinitely long array.

Figure 33b shows the result of a numerical evaluation of (3.52) with $C_0 = 1$. This graph indicates that the center oscillator approaches the injection frequency most rapidly and the others follow at later times, as one would expect. Integrating this frequency function with respect to time yields the phase function, shown in Figure 33a, where the use of $\bar{\omega}$ suppresses the linear time dependence arising from the frequency transition.

---


Phase Dynamics

Now that we have introduced the modeling of COAs, let us extend the theory to explore the interesting features of the phase dynamics, mainly the steady state phase distribution and the transient to reach that steady state after detuning some of the elements. Based on these results, the COA ability of shifterless beamsteering will be presented.

Free-Running Arrays

Steady State Solution

When no injection is present, we can cast the results from the previous chapter

\[
\frac{d\phi_i}{dt} = (\omega_i - \omega_r) + \epsilon \omega_{3dB} \cos \Phi \sum_{j=i-1}^{i+1} \sin(\phi_i - \phi_j) \pm \rho_i \omega_{3dB} \sin(\phi_i - \psi_i) \quad i = 1...N
\]

(4.1)

into a form containing only relative phases defined as
This eliminates the problem of having one arbitrary phase and also reduces the order of the system by one. It also removes the unknown frequency $\omega$, which we can find from any one of equations in (4.1), after solving for the $\Delta\phi_i$.

The phase equations can thus be written in matrix form as

$$\frac{d}{dt}\Delta\phi_i = \Delta\beta_i + \Delta\omega_{lock} \cos \Phi \ As \ i = 1...N$$ (4.3)

where $\Delta\phi$ and $\Delta\beta$ are vectors with elements $\Delta\phi_i$ and $\Delta\beta_i$ and

$$\overline{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ \vdots & \vdots & \vdots \\ 0 & -1 & 2 \end{pmatrix}, \quad \bar{s} = \begin{pmatrix} \sin \Delta\phi_1 \\ \sin \Delta\phi_2 \\ \vdots \\ \sin \Delta\phi_{N-1} \end{pmatrix}$$ (4.4)

Setting the time derivative equal to zero gives an algebraic equation for the steady state phase differences in terms of the oscillator free-running frequencies,

$$\bar{s} = \pm \frac{1}{\Delta\omega_{lock} \cos \Phi} \overline{A}^{-1}\Delta\beta$$ (4.5)

which can be solved by inverting the matrix $\overline{A}$ and solving for the phases using the inverse sine function. The matrix has a simple inverse

$$A^{-1}_{ij} = \frac{j(i-N)}{N} \quad i \geq j \quad A^{-1}_{ij} = A^{-1}_{ji}$$ (4.6)
Clearly, there are no possible solutions of (4.5) if any element of the column vector \( \overrightarrow{A}^{-1} \overrightarrow{\Delta \beta} \) has a magnitude greater than \( \Delta \omega_{lock} |\cos \Phi| \). When there is a valid solution for the sine vector this will correspond to \( 2N - 1 \) different solutions for the phase differences, since the inverse sine function is multi-valued. The correct solution is found by stability analysis.

**Stability of Solution**

Stability is examined by linearizing (4.3) around a steady-state solution. Denoting the perturbation to \( \Delta \phi \) as \( \delta \) gives

\[
\frac{d}{dt} \delta = -M \delta
\]

(4.7)

where the \((N - 1) \times (N - 1)\) stability matrix \( \overrightarrow{M} \) is

\[
\overrightarrow{M} = \pm \Delta \omega_m \cos \Phi \overrightarrow{AC}
\]

(4.8)

and we have defined the diagonal cosine matrix as

\[
\overrightarrow{C} = \begin{pmatrix}
\cos \Delta \phi_1 & 0 \\
0 & \cos \Delta \phi_2 \\
& \ddots \\
0 & \cos \Delta \phi_{N-1}
\end{pmatrix}
\]

(4.9)

A stable mode requires that all the eigenvalues of \( \overrightarrow{M} \) have positive real parts. This will be true of the matrix is positive definite. The matrix \( \overrightarrow{A} \) is always positive.
definite, and the matrix $\mathbf{C}$ is also positive definite when each of the phases lies in the range

$$-90^\circ < \Delta \phi_i < 90^\circ \quad i = 1...N-1 \quad (4.10)$$

Since the product of two positive definite matrices is also positive definite, the eigenvalues of the stability matrix are all real and positive when the phases lie in the above range, as long as $\pm \cos \Phi > 0$. Therefore, (4.10) represents the stable phase region for series oscillators when $\Phi = \pi$, and for parallel oscillators when $\Phi = 2\pi$.

Alternatively, the matrix $\mathbf{C}$ is negative definite when each of the phases lies in the range

$$90^\circ < \Delta \phi_i < 270^\circ \quad i = 1...N-1 \quad (4.11)$$

in which case the eigenvalues of the stability matrix are all real and positive when $\cos \Phi < 0$. Therefore, (4.11) represents the stable phase region for series oscillators when $\Phi = 2\pi$, and for parallel oscillators when $\Phi = \pi$.

<table>
<thead>
<tr>
<th>COUPLING PHASE $\Phi$</th>
<th>PARALLEL VDPO</th>
<th>SERIES VDPO</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2n\pi$</td>
<td>$-90^\circ &lt; \Delta \phi &lt; 90^\circ$</td>
<td>$90^\circ &lt; \Delta \phi &lt; 270^\circ$</td>
</tr>
<tr>
<td>$(2n + 1)\pi$</td>
<td>$90^\circ &lt; \Delta \phi &lt; 270^\circ$</td>
<td>$-90^\circ &lt; \Delta \phi &lt; 90^\circ$</td>
</tr>
</tbody>
</table>
Table 1: Ranges of stable phase shifts for parallel and series VDPO model versus coupling phase.

Note that either phase ranges, (4.10) or (4.11), are sufficient to cause the vector $\vec{s}$ in (4.5) to span all of its possible values, which proves that the stability region fills the entire existence region. Furthermore, over this range of phases the sine functions in $\vec{s}$ are one-to-one. Thus for each set of tunings within the stability region there is a unique phase vector which implies that a unique stable synchronized state exists for a given tuning vector $\Delta \beta$.

**Step Detuning Transient**

Before reaching the steady state, the free-running array passes through a transient period as response to startup or a detuning stimulus. While the discrete model require numerical solution, the continuum model allow for a close form solution of the transitory dynamics. We thus now focus on development of the continuum model and associated boundary conditions, with application to the analysis of time-dependent phase relationships in linear oscillator chains.

One would anticipate that the response time of an array is intimately linked to the size of the array and the locking range; small arrays with large coupling strength would respond more rapidly. This is indeed the case. As we will see, the phases evolve on a time scale that is related to the size of the array and the coupling strength.
These results will reveal their utility in the next session when a description of beam-settling time in a scanning application is obtained, showing that the phase distribution along the array remains “well-behaved” in the sense of maintaining a well-defined beam pattern during the transient period.

A quantitative analysis based on small perturbations has already been described by York\textsuperscript{1}, and yields dominant time-constants on the order of $(L/\pi)^2/\Delta \omega_{lock}$ for small detuning transients, where $L$ is essentially the distance between the detuned oscillator and the farthest edge of the array. Let us now derive analog results for the detuning in the continuum model.

Starting from the previous analysis for the infinitely long array, let us now assume that the array extends from $x = -a$ to $x = a$, thus having oscillators $2a + 1$. The boundary conditions (Neumann) have been derived previously

$$\frac{d\phi}{dx} = 0 \quad \text{for} \quad |x| > a + 1/2 \quad (4.12)$$

Using the same Laplace methods, it can be shown that the inverse Laplace transformation is reduced to residues evaluation, which leads to

$$\phi(x, \tau) = \frac{C\tau}{2a + 1} + C \sum_{n=1}^{\infty} \frac{2 \cos(h\sqrt{\sigma_n}) \cos(x\sqrt{\sigma_n})}{(2a + 1)\sigma_n} \left(1 - e^{-\sigma_n \tau}\right)$$

$$+ C \sum_{m=0}^{\infty} \frac{2 \sin(h\sqrt{\sigma_m}) \sin(x\sqrt{\sigma_m})}{(2a + 1)\sigma_m} \left(1 - e^{-\sigma_m \tau}\right) \quad (4.13)$$

where
The first term exhibits the shift in the ensemble frequency resulting from step detuning the oscillator at \( x = b \) by an amount \( C \) locking ranges as defined previously by

\[
- \frac{\omega_{\text{tune}} - \omega_r}{\Delta \omega_{\text{lock}}} = Cu(\tau) \delta(x - b)
\]  

(4.16)

Since the steady-state ensemble frequency, which is equal to the average of the tuning frequencies, is now no longer equal to the reference frequency, the phase solution includes a linear time dependence, which corresponds to this frequency difference \( C/(2a+1) \). This may also be seen directly from the averaging of the dynamic equation formerly studied

\[
\left\langle \frac{\partial^2 \phi}{\partial x^2} \right\rangle - \frac{\partial \left\langle \phi \right\rangle}{\partial \tau} - \frac{\left\langle \omega_{\text{tune}} \right\rangle - \omega_r}{\Delta \omega_{\text{lock}}} = - \frac{\left\langle \omega_{\text{tune}} \right\rangle - \omega_r}{\Delta \omega_{\text{lock}}}
\]

(4.17)

Recall that the first term is zero because of the Neumann boundary conditions. The right-hand side is a constant equal to the change in the average tuning frequency, i.e. \( C/(2a+1) \). Thus, we have

\[
\frac{d \left\langle \phi \right\rangle}{dt} = \frac{C}{2a+1}
\]

(4.18)
or

$$\langle \phi \rangle = \frac{C \tau}{2a + 1}$$

(4.19)

The constant of integration is zero by virtue of the initial condition that the initial phase is zero. Thus, we can conclude that, aside from this linear term, the average of the phase over the array remains zero for all time. The behavior of an array with 21 oscillators when the oscillator at is detuned by locking ranges is displayed in Figure 34, which is obtained by direct evaluation of (4.13) suppressing the term linear in time. The non-exponential summations are merely Fourier series expressions for the steady-state phase distribution and can be evaluated in closed form leading to

$$\dot{\phi}(x) = C \sum_{n=1}^{\infty} \frac{2 \cos(b \sqrt{\sigma_n}) \cos(x \sqrt{\sigma_n})}{(2a + 1) \sigma_n} + C \sum_{m=0}^{\infty} \frac{2 \sin(b \sqrt{\sigma_m}) \sin(x \sqrt{\sigma_m})}{(2a + 1) \sigma_m} =$$

$$= \frac{C}{2(2a + 1)} \left[ x^2 + b^2 - (2a + 1)|b - x| + \frac{1}{6}(2a + 1)^2 \right]$$

(4.20)

As will be seen shortly, the quadratic steady-state phase is to be expected since it is basically a solution of Poisson’s equation with a constant source term.
Figure 34: Response of a coupled oscillator array under step detuning of the element at $x = 5$.

The slowest exponential decay rate can be found by evaluating (4.15) for $m = 1$. Thus,

$$\sigma_{\text{min}} \approx \left( \frac{\pi}{2a + 1} \right)^2$$ \hspace{1cm} (4.21)

indicating that the response time of the array under detuning of one oscillator is proportional to the square of the number of oscillators (for large $a$). If the center oscillator is detuned, (4.15) is no longer the relevant set of eigenvalues because the corresponding residues are zero by symmetry. In this case, the slowest decay rate is found from the smallest nonzero member of the set (4.14), i.e.,

$$\sigma_{\text{min}} \approx \left( \frac{2\pi}{2a + 1} \right)^2$$ \hspace{1cm} (4.22)

which represents a response time that is twice as fast as the nonsymmetrical case, but, nevertheless, again proportional to the square of the number of oscillators.
(again, for large \(a\)). Similar results are obtained by superposition when multiple oscillators are detuned. The key result is that, in such an arrangement, the slowest time constant governing the dynamics is proportional to the square of the number of elements in the array.

The steady-state solution derived as a limit of the dynamic solution presented above can also be obtained directly without resorting to the Laplace transformation.

Recalling the electrostatic analysis that led to

\[
\frac{d^2 \phi}{dx^2} = -\rho, \quad (4.23)
\]

we can reformulate the steady state phase distribution as the solution, subject to Neumann boundary conditions at the endpoints of the array, to

\[
\frac{d^2 \phi}{dt^2} = \Omega_0 - C \delta (x - b) \quad (4.24)
\]

and it can be shown that the final results is in agreement with (4.20)

\[
\hat{\phi}(x) = \frac{C}{2(2a + 1)} \left[ x^2 + b^2 - (2a + 1)|b - x| + \frac{1}{6}(2a + 1)^2 \right] \quad (4.25)
\]

Recalling now that to maintain lock the phase difference between two adjacent oscillators is limited to \(\pi/2\), which limits the phase gradient, one can show from (4.25) that \(C\) is limited to

\[
C \leq \frac{\pi (2a + 1)}{2(a + |b|)} \quad (4.26)
\]
Recall that $C$ is the amount of detuning measured in locking ranges. In the linearized theory, this would imply, for example, that for a very large array, i.e., for a very large value of $a$, the maximum amount of detuning of a single oscillator would approach $\pi$ locking ranges. However, it is possible to improve on this linear theory estimate by defining an “effective locking range” $\Delta \tilde{\omega}_{\text{lock}}$ as follows. Instead of replacing the sine function with its argument, we rewrite the sine terms in the form

$$\Delta \omega_{\text{lock}} \sin(\Delta \phi) = \Delta \omega_{\text{lock}} \frac{\sin(\Delta \phi)}{\Delta \phi} \Delta \phi = \Delta \tilde{\omega}_{\text{lock}} \Delta \phi \quad (4.27)$$

Now, for small inter-oscillator phase difference $\Delta \phi$, this effective locking range will be approximately equal to $2/\pi$ the true locking range. However, near the limits of lock, it will approach times the true locking range. Thus, the maximum amount of detuning is more accurately represented by the formula

$$\Delta \omega_{\text{max}} = \frac{2a + 1}{a + |b|} \Delta \omega_{\text{lock}} \quad (4.28)$$

instead of (4.26) and for a very large array, this will approach two locking ranges instead of $\pi$ locking ranges. This, of course, is still only an estimate because the solution (4.25) was obtained under the assumption of constant effective locking range when, in fact, the effective locking range defined by (4.27) varies from oscillator pair to oscillator pair for any, but a linear, phase variation. One can say, however, that this represents an upper bound on the detuning for nearest neighbor
coupling because the maximum sum of two sine functions is two, thus, the detuning can never exceed two true locking ranges.

**Scanning by Edge Detuning**

It is well known that the positions of the far-field maxima and minima of a monochromatic array of point sources can be altered by introducing a constant phase shift between neighboring elements. Consequently, manipulation of this phase gradient allows control of the far-field intensity pattern, i.e. beam steering. It will be shown that by suitably adjusting the natural frequencies, beam steering of COA is possible.

The analytical techniques developed above have demonstrated an ability to synthesize certain phase relationships in an oscillator array using a suitable coupling scheme, and by manipulating the free-running frequencies in the array. It is natural to explore the possibility of manipulating the phase for beam control, in which case a constant phase progression is typically desired.

Beam-scanning arrays using coupled oscillators have been developed for several coupling schemes. The chief advantage of this concept is the total elimination of phase shifter circuitry, and in most cases, an elimination of the RF feed network. The disadvantages of this approach are a relatively small bandwidth, limited scanning range, and difficulties in constructing oscillators with highly reproducible characteristics with high yield. However, as we will see in the next
chapters, such problems can be alleviated through advances in oscillator design, the use of phase-locked-loop techniques, and clever circuit augmentations.

From (4.5), we can establish the required conditions for a uniform phase progression $\Delta \phi$ by inserting the $(N-1)$ element sine vector

$$s = \sin \Delta \phi (1,1,\cdots,1)^T$$

which gives the result

$$\cos \sin 1 0 1 1 \cos \sin 1$$

This implies that a uniform phase shift is induced simply by detuning the end-elements of the array (relative to the central elements) by equal amounts and in opposite directions. The amount of detuning establishes the amount of the induced phase shift. Inserting the result back into

$$\frac{d\phi_i}{dt} = (\omega_i - \omega_r) - \sum_{j=i-1}^{i+1} \Delta \omega_{lock,ij} \sin(\Phi_{ij} + \phi_i - \phi_j) - \Delta \omega_{lock,i,ij} \sin(\phi_i - \psi_i) \quad i = 1...N$$

we find that the steady-state synchronized frequency is the same as the free-running frequencies of the central elements, independent of the end-element tuning since the ends are tuned in opposite directions. To summarize, the required frequency distribution is
\[
\omega_i = \begin{cases} 
\omega + \Delta \omega_{\text{lock}} \cos \Phi \sin \Delta \phi & i = 1 \\
\omega & 1 < i < N \\
\omega + \Delta \omega_{\text{lock}} \cos \Phi \sin \Delta \phi & i = N
\end{cases}
\] (4.32)

The range of phase shifts that can be synthesized depends on the oscillator model and coupling network as described in Table 1. We see that end-fire or broadside scanning arrays are possible by proper selection of these parameters.

![Figure 35: Implementation via a Matlab model of the phase dynamic behavior of COAs.](image-url)
Using the modified unit cell model in Figure 35 to account simply for the phase dynamics, we could simulate in Matlab the response of the edge detuning as shown in Figure 36. The small asymmetry of the detuning is justified by the fact that we assumed constant locking range, while at the edges, the change in natural frequencies changes the actual locking range.
One important issue that will be addressed with more details later is the relation between detuning span and speed, a concept related to the ‘slew rate’ of the system. Since two neighbors oscillators cannot have a phase difference larger than $\pi/2$, a step detuning too fast compared to the response of the array can lead the array to unlock. For the case in Figure 36b it is clear the limitation of the initial step difference.

This simple and useful result has the disadvantage of being difficult to explain in a physically appealing manner. Moreover a natural question to ask is, how long does it take for the oscillators to reach an equilibrium phase distribution and a common locked frequency after experiencing a detuning event? For the beam-scanning technique just described, this would relate to the maximum speed with which the beam can be moved from one angular position to another. Also of interest is the behavior of the phase distribution during this transient event. The equivalent continuum models for discrete oscillator chains can provide valuable insight into the behavior of these systems in both the transient and steady state cases.

The complete dynamic solution can be written as the difference of two solutions of the form (4.13). Each of these corresponds to detuning of the form (4.16) with unit step time dependence, one solution with $b = a$ and one with $b = -a$ using antisymmetrical tuning, i.e. $\Delta \omega_L = -\Delta \omega_R = \Delta \omega_{tune}$. The result is
\[
\phi(x, \tau) = \frac{\Delta \omega_{\text{tune}}}{\Delta \omega_{\text{lock}}} C \sum_{m=0}^{\infty} \frac{2 \sin(a \sqrt{\sigma_m}) \sin(x \sqrt{\sigma_m})}{(2a + 1) \sigma_m} \left(1 - e^{-\sigma_m \tau}\right) \quad (4.33)
\]

This function is shown in a and the corresponding far-zone radiation pattern of a 21-element array with half wavelength element spacing driven with these phases is shown in b. Interestingly the pattern retains its basic shape, both main beam and sidelobes, throughout the transient period. The duration of the transient period is governed by the time constant (4.21).
The steady state solution can again be found directly from the electrostatic Poisson’s analogy. When only the end oscillators are detuned, i.e., when

\[
\omega(x) = \omega_0 + \Delta \omega_L \delta(x + a) + \Delta \omega_R \delta(x - a)
\]  

(4.34)

By superposition, the result can be immediately written as the sum of two expressions of the form (4.25) with \( b = a \) and \( b = -a \). The result is

\[
\hat{\phi}(x) = \left( \frac{\Delta \omega_L + \Delta \omega_R}{2 \Delta \omega_{lock}} \right) \left( \frac{x^2 + b^2}{2a + 1} + \frac{2a + 1}{6} \right) - \left( \frac{\Delta \omega_L - \Delta \omega_R}{2 \Delta \omega_{lock}} \right) x
\]  

(4.35)

From this expression, it is clear that the sum of the detunings at the two ends determines the quadratic part of the phase, while the difference determines the linear part of the phase. Thus, equal and opposite detuning of the end oscillators produces a linear steady-state phase distribution in the aperture and, consequently, steers the beam. Of course, any detuning of the end oscillators can be resolved into even and odd parts with the even part controlling the quadratic phase and the odd part controlling the linear phase.

Using this result, we can understand the beam-scanning method easily by analogy to electrostatics. The frequency distribution required to scan the beam is uniform except at the boundaries of the array. This would be analogous to a parallel-plate capacitor where the charge is zero between the plates, and non-zero on the plates with opposite signs on each plate. As we know, the potential is varies.
linearly between the plates of a capacitor, with a slope proportional to the charge. Therefore, in the oscillator problem we expect a linear phase distribution with the phase shift proportional to the frequency tunings at the end of the array.

In this section we have derived a simple relationship between the steady state aperture phase distribution and the free-running frequencies of the oscillators and show that, while essentially any reasonable aperture phase distribution is obtainable with proper tuning, useful distributions are obtainable by tuning only the perimeter oscillators of the array.

**Modulation**

Now we will use the discrete and continuum models to study another cases of interest in antenna engineering, the dynamics of the modulated system under time varying tuning and under time dependent external injection locking.

As we found out from the study of the single VDPO, modulation can be achieved via tuning of the free-running frequency as well as via the injection of an external reference signal.

An important observation can be done a priori. The control of the array can be performed in two ways: globally or locally. Clearly the globally controlled array will respond faster and the overall performance (both modulation and noise) will follow the one of the single oscillator, but the circuitry will become more complex. On the other hand when the control occurs through one or few array elements, the
modulation signal has to ‘flow’ through the array and this means longer response time, even though the implementation can be very simple.

**Modulation Schemes**

Several ideas have been proposed for the modulation of coupled oscillator arrays based on the different coupling topologies and type of modulations. For the most practical (from the microwave engineering point of view) coupling scheme proposed above, clearly FM and PM techniques seem to be feasible.

External locking signals may be applied to one or more array elements in order to frequency- or phase-modulate the system. In a subsequent section, we will also demonstrate advantages of external locking with respect to noise performance. In most applications, we would like to maintain a uniform phase progression in the presence of the externally applied locking signals.

AM is not of practical utility, since, as we showed for the single oscillator, a deep knowledge of the nonlinear amplitude dynamics is needed and for the array in particular this is not yet the case. As we also already stated, one could think to use voltage-controlled attenuator before the radiating elements, but this solution would make control and circuitry impractical.

**Phase Modulation**

Concerning the FM and PM modulation several schemes are available. The considerations for the single VDPO discussed previously are still valid.
For the PM we can image to globally or locally inject the array with a PM source. Or one can imagine that, keeping the source constant and controlling all together the natural frequencies, the array remains locked to the reference signal but with a modulated phase difference with it.

The issue is that examining

\[
\frac{d\phi_i}{dt} = (\omega_i - \omega_r) + \sum_{j \neq i} \frac{\alpha_{ij}}{\alpha_i} e_{ij} \omega_{3dB,j} \sin \left( \Phi_{ij} + \phi_i - \phi_j \right) \mp \frac{\rho_i}{\alpha_i} \omega_{3dB} \sin (\phi_i - \psi_i) \quad i = 1 \ldots N
\]

(4.36)

when \( \rho_i \neq 0 \), we find that the competing effects of injection locking and mutual coupling tend to preclude uniform phase progressions. The problem could be solved if the phasing of the injected signals is identical with that arising from mutual coupling, and so tends to reinforce the desired solution or if only a single array element is injection locked.

These are very general observations; there may be special circumstances where careful adjustment of all the free-running frequencies and phasing of the injected signals may lead to desirable phase distributions. However, these solutions are difficult to quantify analytically due to the nonlinear nature of the equations. Therefore we will consider the following two cases, which appear to have practical merit.
York proposed that each oscillator should feed an antenna, so that injected signals can be applied quasi-optically or via local circuits, and that a parallel oscillator model is used: in this case the incident locking beam and the mutual coupling act in concert to produce an output beam emerging as if specularly reflected. The system thus resembles a quasi-optical injection-locked amplifier.

In this case the global illumination of the array with $\rho_i = \rho$ and $\psi_i - \psi_{i-1} = \Delta \psi$, with the free-running frequencies adjusted so that $\Delta \psi = \Delta \phi$. The injected signal establishes a common phase reference, which we define as $\psi_1 = 0$. If the central array elements are adjusted so that $\omega_i = \omega_{inj}$, then the above assumptions are satisfied by

$$\omega_i = \begin{cases} 
\omega_{inj} + \Delta \omega_{lock} \sin \Delta \phi & i = 1 \\
\omega_{inj} & 1 < i < N \\
\omega_{inj} - \Delta \omega_{lock} \sin \Delta \phi & i = N 
\end{cases} \quad (4.37)$$

which is independent of the locking signal strength, and is the same condition required to establish a uniform phase progression in a mutually synchronized array with no locking. It can be shown that this mode is stable as long under the conditions described by Table 1.

If now we consider a single element of the array locked to an external source with the free-running frequencies adjusted to produce a uniform phase progression $\Delta \phi$. If the $l^{th}$ element is externally locked, we write $\rho_i = \rho \delta_{il}$. Here we can distinguish between external locking of a central array element and an end element.
\((i = 1 \text{ or } i = N/2)\) of the array. In the former case, we find that a valid free-running frequency distribution is given by (4.37) with \(\Delta \psi = \Delta \phi\). When an end-element is locked, we find

\[
\omega_i = \begin{cases} 
\omega_{inj} + \omega_{sDB} \left( \epsilon \sin \Delta \phi - \rho \sin \hat{\phi}_1 \right) & i = 1 \\
\omega_{inj} & 1 < i < N \\
\omega_{inj} - \Delta \omega_{lock} \sin \Delta \phi & i = N
\end{cases}
\] (4.38)

If we choose \(\rho = \epsilon\) and \(\hat{\phi}_1 = \Delta \phi\), then the output beam is controlled only by a single frequency variable, or equivalently, a single DC voltage.

As third and most practical case we assume that the modulation occurs at the tuning ports as this can be done without worrying about matching and line length, being at lower frequencies. Since the edge elements are detuned to obtain the correct beam scanning and thus to preserve the phase difference distribution along the array, all the oscillators need to be modulated simultaneously. Now, when this occurs the average frequency would change and to avoid so, we can have one of the oscillators locked to an external source to force the array to lock to this reference. The center element seems to be the most appropriate, since we already can imagine and will soon prove that the transient evolves in a diffusive way.

The Matlab implementation of such scheme gives the results shown in. As it can be seen, even this method, which is the most practical, has limitations. A large span of the phase modulation will actually also stir the beam since the locking range along the array is not constant, because it depends on the natural frequency.
A way to avoid this problem is to implement in the control circuitry a nonlinear function to compensate these fluctuations during modulation. We will see later that this problem will not occur in CPLLAs.

![Graph](image)

**Figure 38:** Phase modulation while edge detuning of a 5 element COA.

**Frequency Modulation**

For a frequency-modulated injection signal, \( \omega_{\text{inj}} \) varies with time, and hence the phase relationships will fluctuate. Keeping the maximum frequency excursions to a small fraction of the locking range can minimize this. It turns out to be desirable to minimize signal distortion, as shown for a single oscillator.

For simplicity we consider a step change in injection frequency, \( \omega_{\text{inj}}(t) = \omega_o + \Delta \omega \cdot u(t) \) where \( u(t) \) is the unit step function. This is useful in establishing estimates of response time of the system for more complicated
modulation signals. We need to go back to our continuum model to get an insight of such modulation.

To derive the dynamic behavior of the phase in such an array with the element at externally injection locked, we must recall

$$\frac{\partial^2 \phi}{\partial x^2} - C \delta(x-b) \phi - \frac{\partial \phi}{\partial \tau} = -C_0 \delta(x-b) u(\tau)$$  \hspace{1cm} (4.39)

and impose the classical Neumann boundary conditions at the ends of the array and the slope discontinuity condition at the injection point. Laplace method reduces to the estimation of the location of poles of a transcendental equation. The approximate shows that the response time is roughly proportional to the square of the number of elements in the array between the injection point and the farthest end

$$\sigma_{\text{min}} \approx \left[ \frac{\pi}{2(a+b) + 1} \right]^2$$  \hspace{1cm} (4.40)

As in the infinite array case, the inverse transform of the frequency approaches zero at infinite time. However, unlike the infinite case, the approach is exponential instead of $1/\sqrt{\tau}$. Correspondingly, aside from the linear time dependence, the phase approaches a temporal constant at infinite time, but this temporal constant depends on parabolically

$$\lim_{\tau \to \infty} \left\{ \phi(x,\tau) - C_0 \tau \right\} = \frac{C_0}{2} \left[ \left( x^2 - b^2 \right) - (2a+1)(b-x) - \frac{2}{C} \right]$$  \hspace{1cm} (4.41)
This phase difference at the injection point must be small for the present linearized theory to apply. In fact, recalling that this phase difference replaces the sine of this phase difference in the original nonlinear theory, we find that 

\[(2a + 1)C_0\]

must be less than \(C\) to maintain phase lock. Therefore, we arrive at the requirement that

\[C_0 < \frac{C}{2a + 1}\]  

(4.42)

to maintain phase lock in steady state. That is, as can been noted using the nonlinear discrete model, the frequency shift, which can be induced by a step change in the frequency injected at one oscillator of the array is limited, not just to the locking range, but to the locking range divided by the number of oscillators in the array.

Figure 39: Continuum (a) and discrete (b) results for the phases in a COA of 21 oscillators in which oscillator 5 is injection locked to an externally derived signal stepped in frequency.
Figure 39a shows the calculated phase variation, suppressing the linear dependence, for an example in which \( a = 10 \), \( b = 5 \) and \( C \) is unity. According to (4.42), \( C_0 \) must be less than \( 1/21 \). Here, we choose it to be 0.04. As a validation, the same case was computed via Runge–Kutta solution of the nonlinear discrete model equations and the result, shown in Figure 39b, is indistinguishable from that of the continuum formulation. Figure 40 shows the corresponding frequency variation.

If the injection point is located at the center of the array, i.e. \( b = 0 \), then we have the fastest transient, since

\[
\sigma_{\text{min}} \approx \left[ \frac{\pi}{2a+1} \right]^2
\]

(4.43)
\[ \lim_{\tau \to \infty} \{ \phi(x, \tau) - C_0 \tau \} = \frac{C_0}{2} \left[ (x^2 - b^2) - (2a + 1) |b - x| - \frac{2}{C} (2a + 1) \right] \] (4.44)

If now we assume that all the oscillators are externally injected, returning to

\[ \frac{\partial^2 \phi}{\partial x^2} - V(x) \phi - \frac{\partial \phi}{\partial \tau} = -\frac{\omega_{\text{tune}} - \omega_r}{\Delta \omega_{\text{lock}}} - V(x) \psi(\tau) \] (4.45)

and setting \( V(x) = C \) results in

\[ \frac{\partial^2 \phi}{\partial x^2} - C \phi - \frac{\partial \phi}{\partial \tau} = -\frac{\omega_{\text{tune}} - \omega_r}{\Delta \omega_{\text{lock}}} - C \psi(\tau) \] (4.46)

The solution is of the form

\[ \phi(x, \tau) \phi(x, \tau) = C_0 \tau u(\tau) - \frac{C_0}{C} \left( 1 - e^{-C \tau} \right) u(\tau) \] (4.47)

Upon differentiation with respect to time, we find that

\[ \omega(x, \tau) = \omega_0 + C_0 \left( 1 - e^{-C \tau} \right) u(\tau) \] (4.48)

which indicates that the frequency of all the oscillators simultaneously evolves from the initial value \( \omega_0 \) to the final value \( \omega_0 + C_0 \) with a time constant

\[ \frac{C}{\Delta \omega_{\text{lock}}} = \frac{1}{\Delta \omega_{\text{inj}}} \] (4.49)

where is the locking range of the externally injected oscillator in the array. That is, the time constant is proportional to the injection signal locking range rather than
the interoscillator locking range. Moreover, unlike in (4.40), there is no dependence on the number of oscillators in the array.

In conclusion, the response of the array under global excitation is essentially that of a single array element in the absence of mutual coupling, and is observed to response very quickly. In the case of the array with a single locked element, the response of the array is significantly slower and dictated by the number of array elements between the injected element and the array periphery. It is much like the response of the arrays to detuning transients as described earlier. We also showed that the modulation depth, that is, the width of the modulation spectrum, is severely limited in the case of a single locked element to a maximum frequency step of $\Delta\omega_{\text{lock}}/N$.

**Stephan’s Beam Scanning**

As mentioned previously, Stephan proposed the use of the external signal injection as a means of steering the radiated beam. The starting point is to assume that two of the oscillators in the array are injection locked to externally derived signals of the same frequency, but differing phase. Let’s see what happens in case of step and gradual detuning.

**Step Detuning**

Assuming a step change in phase of the injection signals, equation (4.45) becomes
where the $B$’s measure the strengths of the two injection signals, the $b$’s are their locations, and the $p$’s are their phases. The inverse transform can again be found using residue calculus. Here, again, the poles all lie on the negative real axis. Using the final value theorem (residue at the pole $\sigma = 0$), we find that the steady-state phase distribution is

$$\hat{\phi}(x) = \frac{B_2 p_2 + B_1 p_1 + \frac{1}{2} B_1 B_2 \left( (b_2 - b_1)(p_1 + p_2) + (|b_1 - x| - |b_2 - x|)(p_2 - p_1) \right)}{B_2 + B_1 + B_1 B_2 (b_2 - b_1)}$$ (4.51)

Note that there is no constraint corresponding to (4.42) here because the injection signals have the same frequency as the array; they are merely shifted in phase. The phase shift must only be confined to less than $\pi/2$ to maintain lock. This would appear, at first glance, to be a serious drawback associated with this beam-steering technique. However, as will be seen, this limit only applies if the phase is changed stepwise in time. If a gradual phase shift is introduced, the final value is theoretically limited only to $\pi/2$ times the number of oscillators in the array less one. As an example of beam steering, we choose a case where a 21-oscillator array is injection locked at the ends with signals having equal amplitude and antisymmetric phase. That is,
\begin{align*}
B_1 &= B_2 = 1 \\
b_2 &= -b_1 = a \\
p_2 &= -p_1 = 60^\circ
\end{align*}

(4.52)

Figure 41: 21-element COA with step end shift of 120°.

The resulting dynamic behavior of the oscillator phases is shown in a. When this phase distribution is applied to a 21-element linear array of radiating elements separated by a half-wavelength, the resulting steering angle is only about 0.6 beamwidths. Clearly, greater phase shift is needed. This will be addressed below.

**Gradual Phase Shift**

If wide angle scanning is desired, large phase shifts must be produced. This requires a gradual shift of the phase of the injected signals if lock is to be maintained. By convolving the step function with a Gaussian, the transition can be made gradual. The corresponding solution for the phase can be obtained by convolving the step solution with the same Gaussian. Since, in the time domain, the
solution is expressed as a sum of exponentials, one need only convolve each exponential with the Gaussian \( g(\tau) = \exp\left[-\alpha (\tau - \tau_0)^2\right] \). Thus, to obtain the solution for the gradual phase change, one needs only multiply each of the exponentials in the residue series by the above function (which involves the pole location, \(-\sigma_n\)). It can be shown that the above convolution can be written as in terms of complementary error, defined as \( \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt \)

\[
\phi(x, \tau) = \sum_{n=0}^\infty A_n e^{-\sigma_n r} * g(\tau) = \sum_{n=0}^\infty A_n e^{-\sigma_n \tau} \left\{ \frac{e^{-\sigma_n \tau_0} e^{\sigma_n^2/(4\alpha)}}{\sqrt{\pi \alpha}} \left[ \text{erfc}(v_{n1}) - \text{erfc}(v_{n2}) \right] \right\}
\]

(4.53)

where

\[
v_{n1} = -\sqrt{\alpha} \left( \frac{\tau_0 + \sigma_n}{2\alpha} \right)
\]

(4.54)

\[
v_{n2} = \sqrt{\alpha} \left[ \tau - \left( \frac{\tau_0 + \sigma_n}{2\alpha} \right) \right]
\]

Stephan selected a five-element array and plotted the phase evolution of each oscillator as function of time when the end oscillators were injection locked to two externally derived signals 270° out of phase with each other. This results in the phase evolution shown in a, where we choose \( \tau_0 = 6.0 \) and \( \alpha = 0.05 \) for the Gaussian parameters.
Generalizing this to a 21-element array, increasing the phase difference between the injection signals to $1200^\circ$, and adjusting the Gaussian parameter to 0.01 yields the phase distribution shown in b. Note that, with this selection of parameters, the phase difference between adjacent oscillators never exceeds $90^\circ$, thus, lock is maintained throughout the transient period. When the outputs of these oscillators are applied to a linear array of 21 radiating elements separated by a half-wavelength, the resulting far field is shown in, which illustrates the utility of the Stephan scheme in scanning the beam.
Time Delay Considerations

All the previous analysis has been tacitly performed neglecting internal delays of the oscillator and under the assumption that coupling lines between oscillators are merely giving phase contributions, represented in

\[
\frac{d\phi_i}{dt} = (\omega_i - \omega_r) + \varepsilon \omega_{3dB} \sum_{j=i-1}^{i+1} \sin(\Phi + \phi_j) + \rho_i \omega_{3dB} \sin(\phi_i - \psi_j) \quad i = 1...N
\]

(4.55)

by the constant $\Phi$.

As we will see in the next chapter, in real circuits the microwave signal travels within the oscillator and between array elements with lags, which can deteriorate dramatically the COA performance. Unfortunately very little has been explored of
this field, since the mathematics becomes complex and only non-intuitive numerical solutions are available.

However we already pointed out in the first chapter the negative effect that internal oscillator delay have on the performances of the single VDPO. We have shown that to avoid instability and unlocking from the injection source, longer modulation periods and smaller modulation spans are necessary.

Concerning the coupling delay, recently Yeung and Strogatz\textsuperscript{iv} addressed the issue, generalizing the Kuramoto model of coupled oscillators to allow time-delayed interactions. New phenomena are occurring with this improved analysis, including bistability between synchronized and incoherent states, and unsteady solutions with time-dependent order parameters. An exact formula for the stability boundaries of the incoherent and synchronized states was derived, as a function of the delay, in the special case where the oscillators are identical. The experimental implications of the model are discussed for populations of chirping crickets, where the finite speed of sound causes communication delays, but the analysis could be extended to other physical systems such as COAs or CPLLA.

To account for internal and coupling delays, the (4.55) must be modified into

\[
\frac{d\phi_i(t)}{dt} = (\omega_i - \omega_r)
\]

\[
\mp \omega_{3db} \sum_{j=i}^{i+1} \varepsilon_{ij} \sin \left[ \Phi_{ij} + \phi_i (t - T_{ij}^{in}) - \phi_j (t - T_{ij}^{co}) \right] \mp \rho_i \omega_{3db} \sin (\phi_i - \psi_i)
\]

(4.56)
where the internal and coupling delay have been taken into account with $T_{i}^{in}$ and $T_{ij}^{co}$ respectively.

A deep discussion of such issue is not in the scope of this document, but simulations with our Matlab model show in the effects if the coupling delay on the beamsteering and phase modulation ability.

![Graph showing output phases vs. time with various oscillators labeled Osc1 to Osc5.](image)

**Figure 44:** Deterioration of the PM COA in Figure 38 with 10 ns coupling delay.

Clearly these delays limit mostly the modulation span and period, and thus the transient dynamics. If the system does not become instable, the steady state solution (beamsteering) remains unmodified, since time derivatives are null anyway.

From a design point of view, coupling lines needs to be as small as possible: for example in the case of parallel oscillator, even if Table 1 suggests $\Phi = 2n\pi$,
\( n = 0,1 \) should be used. On the other side, the internal delays are merely issues related to the oscillator design itself.

It is important to point out that instability of the system can also arise from the coupling strength and not only the coupling and internal delay. Ram et al.\(^{v,vi}\) showed that as the coupling strength decreases, the system goes from a locked state to mode-locked and quasi-periodic behaviors and finally to chaos.

Clearly COAs have inherently narrow instantaneous frequency response and hence would probably be restricted to narrowband communications. However, it is well suited to broadband FMCW radar or imaging configurations, in which case the output frequency of the array must be linearly swept over a certain bandwidth and this sweep can be enough slow (compared to the delays) to maintain stability.

**Phase Noise**

Till today the phase noise modeling has been carried out only in the discrete fashion: a distributed noise source/admittance can be conceived, but the statistical treatment of the continuum governing equation leads to mathematical difficulties not yet overcome. On the other side the discrete description of the noise in COAs predicted the experimental results both in the case of external injection and the free-running array. Here we strictly follow the study of Chang et al.\(^{vii,vi,viii,ix}\)
General Case Analysis

From the previous study, we see that robust locking favors a low-$Q$ oscillator design, since this implies a large locking range. Unfortunately, low $Q$-factors also imply larger phase noise. External locking to a low-noise source is a possible solution. We will examine the noise characteristics of nearest-neighbor-coupled oscillator arrays in the presence of a noisy injected signal. Only phase noise is considered; amplitude noise and AM-PM noise conversion are assumed negligibly small in comparison to the PM noise in this work. This assumption finds its base on the results found for the single VDPO in the previous chapter.

We assume parallel oscillators coupled using a network like Figure 30

![Parallel coupled VDPOs](image)

Figure 45: Nearest-neighbor parallel coupled parallel VDPOs.

with $\Phi = 2n\pi$. Following the procedure described earlier, we model the internal phase fluctuations in the $i^{th}$ oscillator by a time-varying susceptance $B_{ni}(t)$, which modifies (4.1) as (note that $\rho_i$ is the injection normalized to the steady state amplitude)
\[
\frac{d\phi_i}{dt} = (\omega_i - \omega_r) - \varepsilon \omega_{3dB} \sum_{j \neq i, j = i}^{i+1} \sin(\phi_i - \phi_j) - \omega_{3dB} \rho_i \sin(\phi_i - \psi_i) - \omega_{3dB} B_{ni}(t) \quad i = 1...N
\]

(4.57)

In previous treatments, the analysis was performed using the total instantaneous phase \(\theta\). However, since \(\omega_r\) is deterministic and correspond to just a shift of the ensemble average of the instantaneous phase noise, here we will use \(\phi\) as we did for the single VDPO. This will allow us to simply implement the model in Matlab, observing just the close carrier noise once the carrier has being removed.

\[
\theta_i = \omega_r t + \phi_i \rightarrow \langle \theta_i \rangle = \omega_r t + \langle \phi_i \rangle \\
\sigma_{\theta_i}^2 = \sigma_{\phi_i}^2
\]

(4.58)

Assuming the fluctuations are small, we can linearize (4.57) around a noise-free solution \(\hat{\phi}\), which is a stable solution to when \(B_{ni} = 0\). Writing \(\phi_i = \hat{\phi}_i + \delta\phi_i\), gives

\[
-\frac{1}{\omega_{3dB}} \frac{d\delta\phi_i}{dt} = \varepsilon \sum_{j \neq i}^{i+1} (\delta\phi_i - \delta\phi_j) \cos(\hat{\phi}_i - \hat{\phi}_j) + \rho_i (\delta\phi_i - \delta\psi_{inj}) \cos(\hat{\phi}_i - \hat{\psi}_j) + B_{ni}(t)
\]

(4.59)

Note that all of the injected signals are assumed coherent (derived from the same source). Therefore, all share a common time-dependent fluctuation \(\psi_{inj}\) (we assume that any relative delays in the paths of the injected signals are short compared with the coherence length of the injection source). Taking the Fourier transform and rearranging some terms gives
\[ \tilde{B}_{ni} - \rho_i \tilde{\delta} \psi_{inj} \cos(\hat{\phi}_i - \tilde{\psi}_j) = \]
\[-e \sum_{j=i-1}^{i+1} (\tilde{\delta} \phi_j - \tilde{\phi}_i) \cos(\hat{\phi}_i - \tilde{\phi}_j) - j \frac{\omega}{\omega_{3dB}} \delta \phi_i - \rho_i \tilde{\phi}_i \cos(\hat{\phi}_i - \tilde{\psi}_j) \]  \hspace{1cm} (4.60)

where the tilde (~) denotes a transformed or spectral variable, and \( \omega \) is the noise frequency measured relative to the carrier. This equation can be written in matrix form, using a similar notation as in:

\[ \overline{N} \cdot \overline{\delta \phi} = \overline{B}_n - \tilde{\delta} \psi_{inj} \rho' \]  \hspace{1cm} (4.61)

where

\[ \overline{\delta \phi} = \begin{pmatrix} \tilde{\delta} \phi_1 \\ \tilde{\delta} \phi_2 \\ \vdots \\ \tilde{\delta} \phi_N \end{pmatrix} \quad \overline{B} = \begin{pmatrix} \tilde{B}_{n1} \\ \tilde{B}_{n2} \\ \vdots \\ \tilde{B}_{nN} \end{pmatrix} \quad \rho' = \begin{pmatrix} \rho_1 \cos(\hat{\phi}_1 - \tilde{\psi}_1) \\ \rho_1 \cos(\hat{\phi}_2 - \tilde{\psi}_2) \\ \vdots \\ \rho_1 \cos(\hat{\phi}_N - \tilde{\psi}_N) \end{pmatrix} \]  \hspace{1cm} (4.62)

and \( \overline{N} \) is a matrix with elements \( n_{ij} \) which are related to the steady-state phase distribution and coupling/injection parameters. In the case of nearest neighbor interaction specified above and assuming (to simplify the math) a realistic constant phase progression along the array and constant phase \( (\Delta \hat{\phi} = \hat{\phi}_i - \hat{\phi}_{i-1}, \ \forall i) \), from (4.60) we can write:
\[
\overline{N} = \varepsilon \cos \Delta \phi \begin{pmatrix}
-1 & jx - y_1 & 0 \\
1 & -2 & jx - y_2 & 1 \\
& & \ddots & \ddots \\
0 & & 1 & -1 & jx - y_N
\end{pmatrix}
\]

(4.63)

where we defined

\[
x = \omega / \left( \Delta \omega_{lock} \cos \Delta \phi \right)
\]

and

\[
y_j = \rho_j \cos \left( \hat{\phi}_j - \hat{\psi}_j \right) / \left( \varepsilon \cos \Delta \phi \right).
\]

The inverse of \( \overline{N} \) is not easily expressed for the general case. The phase fluctuations of the individual oscillator are then determined by inverting (4.61)

\[
\overline{\delta \phi} = \overline{P} \cdot \overline{B}_n - \overline{\delta \psi}_{inj} \overline{P} \cdot \overline{\rho}.
\]

(4.64)

where \( \overline{P} = \overline{N}^{-1} \), so that

\[
\overline{\delta \phi}_j = \sum_{j=1}^{N} p_{ij} \overline{B}_{nj} - \overline{\delta \psi}_{inj} \sum_{j=1}^{N} p_{ij} \rho_j.
\]

(4.65)

where \( p_{ij} \) is an element of the matrix \( \overline{P} \).

The combined output of all the array elements is the most important quantity of interest in coupled-oscillator array applications. Assuming that the outputs are combined efficiently (the power combining occurs in free space), the output signal can be written as

\[
V_{out}(t) = A \sum_{i=1}^{N} \cos \left( \omega_i t + \hat{\phi}_i + \delta \phi_i \right)
\]

(4.66)
where the oscillator are synchronized at $\omega_r$ and have identical steady state amplitude and phase distribution $\hat{\phi}_i$. Looking (for simplicity) at the main beam direction where $\hat{\phi}_i = 0$, $\forall i$, with the assumption of small fluctuation, $\delta\phi_i \ll 1$, and recalling $\cos(A) + \cos(B) = 2\cos[(A + B)/2]\cos[(A - B)/2]$, we find that

$$
\cos(\omega, t + \delta\phi_i) + \cos(\omega, t + \delta\phi_j) = 2\cos\left(\omega, t + \frac{\delta\phi_i + \delta\phi_j}{2}\right)\cos\left(\frac{\delta\phi_i - \delta\phi_j}{2}\right)
$$

and thus

$$
V_{out}(t) = N\delta\phi_i \sum_{i=1}^{N} \cos(\omega, t + \delta\phi_i) \quad (4.67)
$$

where

$$
\tilde{\delta\phi}_{tot} = \frac{1}{N} \sum_{i=1}^{N} \tilde{\delta\phi}_i \quad (4.69)
$$

Using (4.65) we can write (4.69) as

$$
\tilde{\delta\phi}_{tot} = \frac{1}{N} \sum_{j=1}^{N} \sum_{i=1}^{N} P_{ij}\tilde{B}_{nj} - \frac{\tilde{w}_{inj}}{N} \sum_{j=1}^{N} \sum_{i=1}^{N} P_{ij}P'_{ij} \quad (4.70)
$$

The power spectral density of the total phase fluctuation (i.e. the phase noise) is computed assuming that the internal noise sources of the oscillators have the same power spectral density but are uncorrelated and also uncorrelated with the injected signal noise. This leads to a total phase noise described by
\[
|\delta \phi_{tot}|^2 = \left[ \frac{\delta \omega}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} p_{ij} \right]^2 - \left[ \frac{\delta \psi_{inj}}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} p_{ij} \rho_j' \right]^2
\] (4.71)

The first term is the contribution from all the internal noise sources, including the effects of the mutual coupling. The second term is the contribution from the noise of the external injection source. For a given coupling network and injection-locking configuration, the task of noise analysis is reduced to that of computing the matrix elements \(p_{ij}\). In some cases these sums can be resolved analytically.

As previously stated, the inverse of \(\bar{N}\) is not easily expressed for the general case, even for relatively simple coupling topologies. However, note that from the relation \(\bar{P} \cdot \bar{N} = \bar{N} \cdot \bar{P} = I\) we can write

\[
\sum_{j=1}^{N} n_{ij} p_{jk} = \delta_{ik} \rightarrow \sum_{i=1}^{N} \sum_{j=1}^{N} n_{ij} p_{jk} = \sum_{j=1}^{N} p_{jk} \left( \sum_{i=1}^{N} n_{ij} \right) = 1
\] (4.72)

By inspection of (4.63) we can easily see that the term in parentheses is the sum of the \(j^{th}\) row, which thus turns out to be:

\[
\sum_{i=1}^{N} n_{ij} = -\frac{j \omega}{\omega_{3dB}} - \rho_j \cos(\phi_j - \psi_j)
\] (4.73)

**Free-Running Array**

When no injection is present, (4.71) becomes

\[
|\delta \phi_{tot}|^2 = \left[ \frac{\delta \omega}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} p_{ij} \right]^2
\] (4.74)
and using the relations stated in (4.72) and (4.73) we can write

\[
\sum_{i=1}^{N} p_{ij} = j \frac{\omega_{3dB}}{\omega} \quad (4.75)
\]

As a consequence from (4.74) we obtain the interesting result

\[
\left| \tilde{\delta \phi}_{\text{tot}} \right|^2 = \frac{1}{N} \left( \frac{\tilde{B}_n}{\omega / \omega_{3dB}} \right)^2 \quad (4.76)
\]

If now we compare this result to the single free-running VDPO (\( \delta \phi_o \)), we can write

\[
\left| \tilde{\delta \phi}_{\text{tot}} \right|^2 = \frac{1}{N} \left| \tilde{\delta \phi}_o \right|^2 = \frac{1}{N} \left| \tilde{\delta \phi}_{\text{uncoupled}} \right|^2 \quad (4.77)
\]

The free-running phase noise is reduced, for all offset frequencies, by a factor of \( N \) compared with a single oscillator. This result has been shown to hold for any reciprocal coupling network. It can be intuitively explained in this way: the power associated to the carrier grows proportionally to \( 1/N^2 \) since the addition of the deterministic signal is coherent, as shown in (4.68), while the power associated to the noise increases as \( 1/N \) because the mutual synchronization does not lead to any significant correlation of the oscillator noises (to first order), as shown in (4.69).

The numerical implementation of (4.76) in Figure 46 shows the general reduction as a function of \( N \) for the set of parameters described in the previous chapter.
Let us now see the effect of the injection from a low noise external source on the noise performances of COAs.

**Globally Injected Array**

Let us now assume that the array is globally illuminated array (all the $y_i$ in (4.63) are the same), where for simplicity we assume $\rho_j = \rho$ and $\phi_j - \psi_j = 2n\pi$.

Substituting these conditions in (4.73), we find

$$\sum_{i=1}^{N} n_{ij} = -\frac{j\omega}{\omega_{3dB}} - \rho$$  \hspace{1cm} (4.78)

and so, from (4.72),

$$\sum_{j=1}^{N} p_{jk} = \frac{-1}{\rho + j\omega/\omega_{3dB}}$$  \hspace{1cm} (4.79)
The total output noise from (4.71) is then

\[ |\tilde{\phi}_{\text{tot}}|^2 = \frac{|\tilde{\phi}_0|^2}{N} \left( \frac{|\omega/\omega_{3dB}|^2}{\rho^2 + (|\omega/\omega_{3dB}|^2)} \right) + |\tilde{\psi}|^2 \left( \frac{\rho^2}{\rho^2 + (|\omega/\omega_{3dB}|^2)} \right) \]  \hspace{1cm} (4.80)

Note that in the absence of an injected signal (\( \rho = 0 \)) (4.80) reduces to (4.76).

Examining (4.80) near the carrier, the output noise is that of the injected signal

\[ \lim_{\omega \to 0} |\tilde{\theta}_{\text{tot}}|^2 = |\tilde{\psi}|^2 \]  \hspace{1cm} (4.81)

and far from the carrier the noise reduces to that of a free-running synchronized array.

The spectral characteristics of the total noise in (4.80) for intermediate frequencies are shown in Figure 47a, for several array sizes, assuming \( \varepsilon = 0.1 \), \( \rho = 0.02 \), and both the internal and injected noise characteristics follow the ideal \( 1/f^2 \) dependence. The individual array elements were taken to have a single sideband noise of -60 dBc/Hz @ 100 kHz offset, typical of a low-\( Q \) microstrip MESFET oscillator. The injection source was modeled by a similar noise source with -130 dBc/Hz @100 kHz offset. Note that the noise characteristics improve slightly with increasing array size, which is due to the \( 1/N \) reduction of the contribution from the internal noise sources, arising from mutual coupling in the array. The dependence on the injected signal strength for the same parameters described above is also shown in Figure 47b.
Following a similar analysis for the case of a single array element coupled to an external locking source, we find, for a signal applied to the \( l \)th element

\[
\sum_{i=1}^{N} n_{ij} = -\frac{j\omega}{\omega_{3dB}} - \rho \delta_{ij}
\]  

(4.82)

which gives, from (4.72),

\[
-\frac{j\omega}{\omega_{3dB}} \sum_{j=1}^{N} p_{jk} - \rho \sum_{j=1}^{N} p_{lk} = 1 \quad k = 1 \ldots N
\]

(4.83)

The total noise is then given by
where the property $p_{ij} = p_{ji}$ was used. We cannot evaluate this expression analytically without first finding the elements of the $l^{th}$ row or column of $\bar{P}$. However, we can examine the limiting behavior near and far from the carrier. Near the carrier, (4.83) gives

$$p_{lk} \approx -\frac{1}{\rho}$$

(4.85)

and substituting (4.84) gives

$$\lim_{\omega \to 0} \left| \delta \theta_{tot} \right|^2 = \left| \bar{\psi} \right|^2$$

(4.86)

Far from the carrier we find

$$\sum_{i=1}^{N} p_{ij} = -\frac{j \omega}{\omega_{3dB}}$$

(4.87)

which gives

$$\lim_{\omega \to \infty} \left| \delta \theta_{tot} \right|^2 = \frac{\left| \delta \theta \right|^2}{N}$$

(4.88)

These are the same asymptotic values as derived for the globally injected case.

The behavior for intermediate frequencies is more complicated computationally but qualitatively similar with respect to frequency. Figure 48a illustrates the total
phase noise as a function of the offset frequency for several different array sizes, with the injection signal applied to the center element. The same noise parameters were used as in Figure 47, but slightly larger injected signal strength was used ($\rho = 0.1$).

**Figure 48: Theoretical PSD of the locally injected array for several sizes (a) and strengths (b).**

Here we observe a significant degradation in the output phase noise with increasing array size. Figure 48b illustrates the dependence on the injection strength. If the injection signal is instead applied at the first array element, the noise characteristics degrade more rapidly with increasing array size. An analysis of the individual noise fluctuations on the array confirms that the individual contributions from the array elements increase with distance from the injected signal, at a rate...
that depends on the coupling strength. Thus, a practical system would clearly favor a large inter-oscillator coupling strength, large injected signal strength, and the injected signal applied at the center of the array.

**Comments**

As it may appear from the previous analysis, one of the limitations of COAs is the limited range of phase shifts that can be synthesized. This could be improved by introducing a frequency doubler circuit after each oscillator, as suggested by Alexanian et al., which effectively doubles the inter-element phase shift. Despite additional circuit complexity, this technique has some additional benefits: the oscillators can be designed at a lower frequency (half the desired output frequency), which is useful because oscillators are sensitive to parasitic reactances, and also because oscillator design is simpler when the device has high gain, which is more easily achieved at lower frequencies. The range of oscillator tuning required to achieve a given scan range is also significantly reduced, which is advantageous since operation of the array near the locking band edge is undesirable due to increased phase noise (since the oscillator noise is also doubled), reduced modulation range, and increased sensitivity to environmental disturbances.

Earlier we only presented the transmitting operation of COAs. This is actually not a limitation of such systems. The edge-detuned scanning configuration can also be employed in a receiving application. This is accomplished by using the
scanning oscillator array as the local oscillator for a set of mixers. It may be possible to merge the transmit and receive functions, especially for FMCW imaging arrays, by making each array element a self-contained FMCW transmitter and receiver, with each array element coupled to its neighbors. Alternatively, each array element could be a self-oscillating mixer. These concepts have not yet been tested.

It is now time to see how the previously described properties can be implemented and which tradeoffs and difficulties the design, fabrication and characterization of COAs face.

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Chapter 5

Coupled Oscillator Arrays: Design and Characterization

The previous theoretical analysis of microwave coupled oscillator arrays need to find experimental verification. The conditions and the tradeoffs behind the assumptions we made are helpful in the design process and set reference criteria for the characterization of the array. In this chapter we will follow the implementation process of a typical COA, starting from the single cell and the coupling network.

Two were our goals. From one side, we defined a set of design rules for the engineer who wants to use off-shelf components to build a COA. In the following, we present the design, implementation and characterization of the single oscillator, appropriate coupling network and control system. Second we implemented a system able to follow the phase and amplitude dynamics of the array without affecting its behavior.

The measurements confirm the theory presented before, while they highlight the benefits and limitations of COA.
**The Single Oscillator**

We separated the project in three main areas of focus: the single oscillator, the coupling network and the control system. The three modules can be designed, implemented and characterized separately.

**Design & Realization**

The single oscillator is the most critical component in the array: noise, cost, stability, frequency control, locking range and size of the whole array are mainly determined by the properties of the single element.

The mutual synchronization is provided by the coupling network, on which single mode (fundamental) operation and modulation bandwidth (or locking range) depends.

Finally due to the large number of oscillators, the control of the system becomes difficult and thus the need for a convenient user interface arises. The interface should facilitate the control of modulation and scanning. An antenna array was designed as radiation load.

To simplify the characterization and since the equivalent model of the oscillator affects the coupling structure, the design of COA should be performed in blocks starting from the single element implementation.

Because past experience showed that the design of the single oscillator from basic active elements (such FETs) is not an easy task because of the need of
matching and isolation, we decided to use off-shelf components and design the embedding resonant circuit. The operating frequency is 2.5 GHz, as several components are available because of the standard wireless LAN.

The array was designed to have a broadside radiation pattern, with relatively large locking range (possibly ~ 100 MHz), compact size and scanning angle of ± 20°.

The compact size and the low price lead us to use surface mount technology (nowadays also high performance inductors with 5 % tolerance are available).

Based on previous design experience, the active device and embedding resonant circuitry was chosen on the following set of specifications:

- Reference Impedance: 50 Ω
- Varactor Frequency Tuning Range: ~ 1 GHz
- Injection Locking Range: ~ 10 MHz @ P_{inj} = 0 dBm
- Output Power: ~ 13 dBm in free oscillation
- Low εᵣ substrate (H = 31 mils, εᵣ = 2.55, copper metalization)

The design of the single oscillator has been carried on with the following objectives:

- Investigate the features of commercial available low-cost active devices and choose the most appropriate for our application
• Design the embedding circuit most suitable for the proper operation of the active device

• Find an appropriate circuit model for the oscillator and determine the values of the equivalent circuit elements

• Determine the characteristic parameters of the injection locked oscillator and associate them with the equivalent circuit

• Propose a suitable coupling network and control system based on the previous considerations

In the microwave literature, there are several circuit configurations suitable for an oscillator. However, in a COA a simple control of the natural oscillation frequency is fundamental. Thus, the most convenient architecture for our project is - at our knowledge - the Voltage Controlled Oscillator (VCO). An active device is embedded in a positive feedback circuit in which the resonance frequency is perturbed (or fully determined) by a voltage-controlled capacitor (a varactor - for instance a reverse biased diode – usually has a tuning voltage ranging 0 - 20 V).)

In choosing the active device, we compared the electrical characteristics of IC oscillators from Pacific Monolithics and Mini-Circuits. At our knowledge these are the two companies that have the widest choice for 1-10 GHz IC oscillators. The PM2503\textsuperscript{ii} RFIC oscillator appears to have the electrical characteristics most suitable for our purposes. We chose to avoid a custom-designed hybrid VCO using low-cost discrete MESFETs for the following reasons:
1) Although the simulations suggested higher output power and wider tuning and locking ranges for the MESFET, the noise in a hybrid circuit can compromise a lot the performance of the active device.

2) Moreover, the IC oscillators offer in general an output buffer that reduces the effects of a non-ideal load on the oscillator behavior. This simplifies the whole process avoiding the design of a buffer stage when using a transistor.

The PM2503 presents also a relatively high output power with a low supply voltage (+5 V). The currents are lower and thus the single oscillator efficiency is higher compared to other available VCOs. The overall output power of the array is determined by the one of the single oscillator by mean of the array efficiency $\eta_A$, function of the combining and oscillator efficiencies:

$$P_{A_{Out}} = \eta_C P_{Out}, \text{ where } \eta_C = \frac{\eta_A}{\eta_{Osc}}$$

$$\eta_{Osc} = \frac{P_{Out}}{P_{DC}} \approx 21\%, \text{ for the PM2503}$$

Once the choice of the active device has been made, the embedding resonator must be designed. The architecture suggested by the datasheet cannot be applied in the present case: we need to reduce to the minimum the length of the microstrip segments. In this way, parasitic inductances, which can shift the center frequency and reduce the tuning range, are avoided. Most important, reducing line lengths, we reduce the side effects the internal delays.
We first study the PM5103 RFIC to determine its equivalent model and the most appropriate approximation to a Van der Pol type oscillator and its characteristic parameters. Concerning the type of resonator, it is important that a single mode of operation (series or parallel) dominates over the other within the frequency of detuning. Otherwise an abrupt change in phase can occur during the detuning and the radiation pattern can degenerate.

From the manuactory datasheet, we extracted the S-parameters of the PM2503 for the input (or coupling) port and we used them to simulate the frequency behavior of the oscillator as function of the varactor bias voltage. It turned out to be close to a circuit with an inductance of 3 nH in series with a capacitance of 0.7 pF. The serial resistance varies due to the nonlinear gain of the active device and we assumed a value of $-30 \, \Omega$ in the range 2.3-2.5 GHz. However if the embedding circuitry has a parallel resonance, then the overall behavior is similar to a parallel VDPO.

In Figure 49 we plotted the results obtained in the simulation for different values of the varactor voltage. The oscillation will take place when the imaginary part of the input impedance goes to zero. In this case, the RFIC seems to be well modeled with a series resonator.

To achieve the maximum performance and stability inherent in each MMIC VCO, it is essential that a good RF ground be established and maintained between the VCO substrate and the mounting surface. The diameter of via holes should be
carefully considered: a simple rule of thumb equation is to use a via diameter equal to the board thickness as not to generate parasitic inductances.

Sufficient decoupling for the input supply voltage, a clean tuning supply voltage, and proper output isolation are also critical parameters needing to be addressed. In particular noise on the dc input voltage supply can AM and FM modulate the output of the VCO. Therefore the supply should be adequately decoupled to reduce as much supply noise as possible. Capacitor values of 100 pF, 0.1 uF, and 10 uF are placed as close as possible to the bias contact and along the supply line of the final assembly. Additional improvement can be gained by isolating the VCO from an unregulated primary power source by the use of voltage regulators and simple R-C filtering at the output of the supply.

Matching the output of the VCO to external circuitry using a 10 dB pad located as close as possible to the output of the device will improve isolation and minimize an impedance mismatch. Lower attenuation levels may be used; however, this will result in decreased load isolation. In our case the internal MMIC circuitry provides a first isolation and matching, but usually standard RF design practices should be followed regarding the use of a 50 Ω microstrip line coupling the RF Out to the 10 dB pad.

The realization of the circuit involves the mask design, the copper etching and the components soldering. During the realization of the circuit, we tried to minimize the effects of the soldering for the same reason previously presented. We
used large inductors (1 mH) to isolate the DC bias circuitry. Ceramic capacitors are used in the bias circuitry to reduce the noise coming from the DC supplies (RF filtering).

Characterization

Once the device model was determined, we designed and realized a test board to characterize the IC with a serial external resonant network. We determined locking range, $Q$-factor, detuning range and nonlinear parameter $\mu$.

To be able to evaluate the injection performances of the oscillator together with its output characteristics, we design the single oscillator layout as a two-port device. The injection and output microstrips have a phase shift of 180° and 90°, respectively, to ease the connection with the measurement instrumentation and the coupling network in the array. Figure 50 shows the UCSB and JPL single cell designs.
We build three oscillators: a parallel, a series and one without feedback to compare their performances and extract the characteristic of the oscillator to explain such performances. Although we performed the measurements for the parallel and the series, here we will present only the results of the series oscillator. The parallel type presents a wider tuning range, but also a larger amplitude variation within this range. Nevertheless, the overall theory and results are similar.

There are few ‘figures of merit’ for the oscillator and they can be deduced from the plots that follow. Based on the results of the previous chapters, the useful equations that allows to determine in a practical way the VDPO parameters are

\[
\Delta \omega_{\text{inj}} = \frac{V_{\text{inj}}}{V_0} \frac{\omega_0}{2 \cdot Q} = \frac{\omega_0}{2 \cdot Q} \sqrt{0.001 \cdot Z_0} \cdot 10^{\frac{P_m[\text{dBm}]}{20}}
\]  

\[(5.2)\]
\[ V_{\text{out}} = V_0 \left( 1 + \frac{\delta}{2 \cdot \mu} \right) - \frac{\delta \cdot V_{\text{inj}}}{2 \cdot \mu} \cdot \cos \Delta \vartheta_{\text{inj}} \] (5.3)

The first parameter we are interested in is the tuning range. Figure 50 shows our results. In the resonant embedding circuit, we preferred to use an inductance of 1.8 nH. Even if the power has a larger variation inside the band, the tuning range is much wider.

The measured results slightly differ from the simulation. We did not take into account that the varactor capacitance experiences a saturation effect with the increasing voltage. This could explain part of the discordance.

![Output Powers vs Output Frequency](image)

Figure 51: Measured Output Power vs. Output Frequency for the Single Oscillator.

We didn’t notice any jump or hysteresis phenomenon in the single oscillator.

Another important parameter is the locking range. It depends on the amplitude of the injection signal \( V_{\text{inj}} = \sqrt{0.001 \cdot 50} = 224mV \) and it is related to the \( Q \) of the
oscillator. We measured at 2.5 GHz the variation of the locking range as a function of the injected power (Figure 52). Thus, using a fitting curve we were able to extract the value of \(Q\). As example, the series oscillator in Figure 51 has a \(Q\) of 21, while the locking range is approximately 10 MHz at 0 dBm injection power:

\[
V_0 = \sqrt{0.001 \cdot Z_0 \cdot 10^{-20}} = \sqrt{0.001 \cdot 50 \cdot 10^{-20}} V = 1.27V
\] (5.4)

\[
\frac{F_0}{2 \cdot Q} \frac{0.001 \cdot Z_0}{V_0} = 10.51 \times 10^6 MHz \rightarrow Q = \frac{F_0}{2 \cdot 10.51 \cdot 1.27} = 20.9
\] (5.5)

When we increase the natural oscillation frequency, keeping \(P_{\text{inj}}\) constant, the locking range extends like.

The value calculated with the network analyzer (1.5 MHz @ -10 dBm) (Figure 52) differs from the one we measured with the spectrum analyzer (3 MHz @ -10 dBm).
dBm). In fact, we don’t know the real power delivered by the network analyzer and, besides, the two instruments do not present the same load to the device, so reflections play a main role in this discrepancy. However this will not affect drastically the array performance.

There is still one very important parameter to extract from our measurements: the non-linearity factor \( \mu \). Following (5.3), we could calculate \( \mu \), using the output power when the oscillator is locked to an external signal (we set \( P_{\text{inj}} = 0 \text{ dBm} \) and \( \delta = 1 \)) and again a fitting formula (Figure 53) \(^v\)

\[
V_{\text{out}} = V_0 \left( 1 + \frac{1}{2 \cdot \mu} \right) - \frac{1}{2} \cdot \frac{0.224}{\mu} \cdot \cos \Delta \theta_{\text{inj}}
\]  

(5.6)

![Figure 53: Experimental calculation of the VDPO nonlinearity factor.](image)

We also note that the oscillator has another useful property: as we increase slightly the bias voltage, the center frequency increases too. This effect can be used,
together with the varactor bias, to set the oscillators inside an array to the same natural oscillation frequency, just barely varying the biasing resistors. In this way, we optimize the coupled locking range dynamics against the spread in the tolerance range.

The RF outputs that will be connected to the radiating elements are isolated from the oscillators (and their resonators) by buffer amplifiers intrinsic to the monolithic microwave integrated circuits (MMIC’s) that were used. Thus, the element spacing is independent of the coupling and can be chosen for optimum performance of the radiating aperture. This is the primary difference between our work and that of our predecessors.

**Coupling Network**

As we have seen in the previous chapters, the design of a broadband coupling network is not a difficult task: a simple transmission line and matching resistors (Figure 54). The length is very important since the stability points for synchronization are based on the VDPO model and the desired beam type (endfire or broadside). The standard microstrip is at our knowledge the easiest coupling network to build and to model.
Using the selected VCO, a five-element array was build at UCSB and a seven-element was fabricated at JPL\textsuperscript{vi}. These arrays are shown in Figure 55 while an enlargement of the center oscillator was shown in Figure 53. Both array were designed to be broadside.

Figure 55: UCSB (left) and JPL (right) realization of the coupled oscillator array.
A partial schematic is illustrated in Figure 56, which includes an end oscillator and the oscillator adjacent to it. The rest of the circuit continues in a like manner to the far end. The theoretical treatment is based on the assumption that the $Q$ of the coupling circuit is much lower than that of the oscillators. To achieve this in our implementation, we terminated the 100- coupling lines using 100- chip resistors, as shown in Figure 56. This was intended to reduce reflections from the ends of the line so that energy storage on the line in the form of standing wave was minimized. Minimum energy storage implies minimal $Q$ (for a fixed loss). However, the actual termination of the line is modified by the value of the series coupling resistor, which is, in turn, set by the desired coupling strength. Moreover, $Q$ is a function of the resistive loss as well as stored energy in the line. Thus, a matched termination does not necessarily correspond to minimal $Q$. Nevertheless, the present arrangement lowers the $Q$ of the coupling line by roughly two orders of magnitude, which is considered sufficient for applicability of the theory. A last observation: from the oscillator point of view, the two identical coupling lines are equivalent to one line with the half value resistors and half $Z_0$ line with the same electrical (not physical!) length.
It has been established that, for the parallel resonance of the present MMIC, the appropriate coupling line length is one wavelength, while the end elements have a corresponding half wavelength to satisfy the Neumann boundary conditions. However, because of the unknown phase shift inherent in connecting the lines to the MMIC, an optimization of the line length was needed.

At UCSB several test board were build to find this optimized length. At JPL it was found useful to arrange for variable line length. Therefore, the inter-connecting (coupling) lines were meandered in a U shape such that they could easily be made to be full wavelength or half-wavelength or anything in between by moving the shorting bar along the U-shaped portion. With detailed measurements and modeling it should be possible to ascertain the proper line length and fabricate it without
adjustment, and this may ultimately be the preferred method of manufacture. However, this was beyond the scope of the present effort. The U-shaped lines were a simpler solution for the present experimental array. One might be concerned about the presence of two parallel lines over a portion of the path, but, at a single frequency, this is equivalent to a single line cut and terminated so as to have the same scattering parameters. Once the optimal positions for the shorting bars were determined, the original lines were cut at each end of the bars to remove the excess line. Technically, this cutting operation changed the coupling phase to a slightly nonoptimal value, but the array locking range remained quite acceptable. Two discontinuities are introduced in each line by the difference in impedance between the lines and shorting bar, and this affects the impedance match at the oscillators. However, as discussed earlier, this match was not optimal in the present array regardless. Figure 57 illustrates a representative portion of the seven-element array. Note that the coupling lines have a gap halfway between the oscillators. This was to provide for a blocking capacitor.

At JPL, the line-length adjustment via shorting bars permitted two theoretically predicted behaviors to be observed and, thus, confirmed during this investigation. First, when the coupling lines are an odd multiple of a quarter-wavelength in effective length, the locking range of the array is reduced to zero and no locking is possible. Second, when the coupling lines are an even multiple of a quarter-wavelength in effective length, the ensemble frequency is equal to the tuning frequency of the individual oscillators, whereas it differs from this value for other
line lengths. This difference is maximized at odd multiples of a quarter-wavelength. Pogorzelski et al.\textsuperscript{vii} were able to interpret these phenomena theoretically.

![Figure 57](image)

**Figure 57:** Partial layout of the JPL seven-element oscillator array using the variable length lines.

Both layouts include a printed common ground interconnect trace beneath the MMIC with several through connections to the underside ground plane. Improving the grounding reduces the parasitic resonances (and thus frequency jumps) in the circuits and together with the matching resistor as termination of the coupling network avoids the presence of hysteresis as shown by Kurokawa\textsuperscript{viii}. Kurokawa describes the jumps and the hysteresis as results of a faraway reflection that creates loops in the Smith Chart of the load reflection coefficient. The characteristic
oscillator line (which depends on the amplitude) crosses the load line when the oscillation takes place. If loops are present, different paths can be taken as the frequency sweeps and the stable points may not follow closely (jumps) and be sensitive to the direction of the sweep (hysteresis).

As final note it is important to distinguish from the outset between arrays of oscillators with strong coupling to nearest neighbors such as those considered by Nogi et al.\textsuperscript{15} and the present array of oscillators with weak coupling to nearest neighbors. In the strong-coupling regime, the neighboring oscillators influence both the phase and amplitude of the oscillation, leading to ensemble modes with both phase and amplitude variations across the array.

In the weakly coupling regime, the lowest order mode dominates because the dynamic behavior of the amplitude is dominated by the saturation of the oscillator and is not significantly influenced by the signals coupled from the neighboring oscillators. Thus, in the present weak-coupled case, as indicated by York\textsuperscript{x}, the amplitudes of the oscillators will remain close to their free running values and the phase can be controlled via tuning of the end oscillators.

**Array Control**

In order to control the array natural frequencies and, as we will see later, perform automatic measurements of phase and amplitude distributions, we designed a control board with the following specifications
• Control of the Voltage on the VCOs
• Up to 0.1 ms Refreshing Time
• Voltage Hold = 0.001 V/s
• Variable Gain for Tuning Range Limiting
• Offset DC Correction for Accurate Control
• LEDs Monitor Channel Saturation and Activation

The board is also used to monitor the calibration and measurements boards through a set of LEDs. The board can be controlled with a maximum refreshing speed of 1 KHz. It creates an interface between hardware and software, so the characterization system is fully computer controlled (through Analog/Digital PCI1200 and GPIB cards)
Figure 58: Control board for the control of the array and the measurement system.

Figure 28 shows the calibration board while Figure 59 shows the program we wrote to control the natural frequency distribution along the array.

At JPL a common tuning voltage bus was used so that only a single precision tuning supply was needed for test. Tuning of the individual oscillators was accomplished via ten-turn potentiometers connecting each oscillator to the bus.

Figure 59: C++ software that controls the whole system.
The automated TTL switch control is followed by sample and hold circuitry to maintain the desired tuning voltage. The typical tuning curves obtained with such measurement system are shown in Figure 60.

**Design and Fabrication of the Radiating Structure**

Each oscillator in the array provides an output RF signal properly phased with respect to the others. To radiate this signal, one must provide each oscillator with a properly designed radiating antenna element. Each element must be properly impedance matched to the oscillator and must radiate the RF energy with a wide enough beamwidth to achieve wide-angle beam-scanning capability. The overall array antenna beam will then be a result of the spatial power combining that takes place in the radiating aperture.
We designed and built an array of radiating element. For the sake of simplicity, we used patch antennas with recessed feeds to improve the input matching.

A microstrip patch was selected as the radiator because of its low profile and small weight, as well as its capacity to be conformally mounted onto a curved surface. This choice also provides negligible mutual coupling between the elements, a property that was confirmed by the agreement between the measured array pattern and that predicted theoretically neglecting mutual coupling. A practical advantage of the microstrip patch is that all array elements can be fabricated on a single slab of substrate material with a single chemical etching process, which can significantly lower the production time and cost.

![Figure 61: Single element and array of patch antennas used as radiating structure.](image)
The configuration of a single microstrip patch element is shown in Figure 61, where a square metallic patch is constructed, by a conventional chemical etching process, on a thin dielectric substrate with a conducting ground plane situated beneath it. The square patch was designed to resonate at 2.50 GHz. It is designed by using Momentum, part of the Ads software package, which employs an integral-equation technique (method of moments). To have a comfortable bandwidth for the radiator, the dielectric substrate was designed to have a thickness of 0.16 cm with a relative dielectric constant of 2.2. This dielectric substrate is made of the Rogers Duroid 5880 impregnated fiber-glass material.

To increase the matching as well as the bandwidth, we used a recessed feed type, for which the impedance can be calculated using the equations defined by Balanis\textsuperscript{xi}.

\[
\Gamma_{in} = \quad (5.7)
\]
The measured radiation pattern of the single-patch element in the $H$-planes and the return loss are shown in Figure 62, in close agreement with published results and where relatively wide beamwidths are demonstrated. In the same figure, the calculated input impedance match, in terms of return loss, is also given. It shows that the antenna resonates at 2.5 GHz and has a 2:1 VSWR (10 dB return loss) bandwidth of about 40 MHz (1.4%).
A five-element array achieves sufficiently narrow beamwidth to demonstrate the beam scanning with adequate beam resolution. Photographs of the actual fabricated array are shown in Figure 61. The five identical elements are spaced uniformly with half free-space wavelength spacing at 2.50 GHz. The elements are arrayed in the $H$-plane. By using the conventional power divider, the radiation performance of the array can be independently assessed without including the effect of the oscillator circuits. These results indicate that the five-element microstrip array developed here is adequate for the oscillator array demonstration.

To design correctly the patch array, equal electrical length of the feeding lines was needed. A program in Mathematica was written to optimize the curving in order to maintain continuity of the first and second derivative while minimizing the total length. In Figure 63 the general criteria used for this design are reported. This program was also used to generate the lines for the calibration and other test boards.
Criteria to calculate the arcs necessary to complete an electrical distance $L$ using curved paths of width $W$, projected cord $D$, height $H$ and angle $\phi$.

Until $L_{Tot} < 2 \pi (3W)$ sections of 4 arcs of $R_{Min} = 3W$ and $\phi = \pi/4$, then 4 arcs with $R$ using the remaining $D$ & $L$.

Note that $\phi_{Max} \leq \pi/4$ & $L/D_{Max} \leq \pi$, for $R = 3W$ & $\phi \leq \pi/4$.

Figure 63: General criteria used to design the equal length feeding lines.

**Measurement of the phase and amplitude distribution**

The most important quantity for a coherently radiating system (such as linear & nonlinear antennas) is the relative phase between elements. An accurate control of the phase distribution and its time evolution is the main factor in the design of coupled oscillators, where the injection locking phenomenon causes the generation of a single frequency radiation with phase distribution along the array determined only by the natural frequencies before locking.

When the number of oscillators is more than two, the far field measurement is the only easy way to approximately know of the phase progression on the array. A near-field measurement can give a better insight of the situation, but the signal
collected by the antenna is generally very small and thus difficult to mix down to obtain the phase. Moreover, in the design of the near-field probe, efficiency, matching, modeling and calibration present difficulties.

For system up to 10 GHz, another method offers relatively easy implementation and result interpretation: the ‘on-board’ phase measurement. The system requires along with the oscillator and antenna arrays, the phase measurement panel, a calibration board and a circuit to control the natural frequencies, to switch between oscillators in the phase measurement and to select the path length in the calibration board. The control board will also regulate the bias of oscillator, control and calibration boards. A computer runs the system with an analog/digital link and collects the results through a GPIB network (Figure 64).

![Schematics of the measurement setup.](image)

*Figure 64: Schematics of the measurement setup.*
The idea is simple: an isolator offers to the oscillator a relatively constant matched load, and a following power splitter divides the signal between the antenna and a switch where also all the other $N-1$ oscillator are collected. The output of the switch is then split in two parts that will be mixed one with a reference signal (one of the oscillators) and one with its $90^\circ$ conjugate. In this way, we can obtain sinus and cosines of the phase between the reference and one of the other elements.

The proposed setup allows ‘on board’ measurements and far field radiation pattern collection at the same time.

Let us now illustrate the process of design, realization and analysis of the five-section system (oscillator array, control network, phase detector, antenna array, calibration board) used to corroborate previous theoretical results, in particular the steady state phase and amplitude distributions.

**Theory and Implementation**

The center element is used as the phase reference. We used isolator to offer to the active elements a constant load.

As shown in Figure 65, the signal of all the oscillators is split in two directions. Half of the signal goes to the radiating element (or to a power meter or to a spectrum analyzer), while the other half is used for the ‘on board’ detection.
The center signal is connected to a hybrid coupler that splits the signal into two and delays one side by 90° with respect to the other. The N-1 oscillators are connected to a common switch, so that we can choose which one is ‘ON’ and leave the others terminated on a 50-Ω load. The output of the switch is then divided into two parts. The two halves of the signal are then mixed with the 0° and 90° phased outputs of the hybrid coupler. Since the two incoming signals are at the same
frequency, the mixer outputs will have an RF and a DC component. The RF component is then filtered out (or absorbed in a 50-W load) while the DC component is measured. The DC component is linearly related to the cosines and sinus of the phase difference between the incoming signals. It also contains the information on the amplitude product. That is,

\[
\begin{align*}
V_{\text{Out},0^\circ} &= \frac{V_3 V_j}{2} \sin(\Delta \phi_j) + RF \\
V_{\text{Out},90^\circ} &= \frac{V_3 V_j}{2} \cos(\Delta \phi_j) + R
\end{align*}
\]  

(5.8)

where \( V_{\text{Out},0^\circ,90^\circ} \) are the outputs of the two mixers, \( V_{3,j} \) the mixer inputs from the center and the \( j \) side element and their phase difference \( \Delta \phi_j \).

We chose to work with the signal amplitude values (not the rms and not the peak-to-peak values) to simplify the math.

The hybrid coupler and the two mixers could be also integrated in a quadrature modulator. We actually used both architectures and verified similar results.

The oscillators output are thus matched and isolated as shown in Figure 67.
Figure 67: Input matching and load isolation of the measurement system.

The system needs a calibration, for two main reasons. First, an ideal mixer has an offset and a phase coefficient (slope of the input-output phase relation) and a non-flat amplitude-phase response (that also depends on frequency). Second the phases and attenuations along the channels are frequency dependent.

Nevertheless, since most commercially available mixers are relatively stable and linear (in phase), the calibration process avoids systematic errors.
Calibration Procedure

To calibrate the system, we designed a calibration board with a 5-way power divider (Figure 68). The centerline is used as input of a 4-way switch that allows us to choose between 4 paths of different length (each with a 90° electrical length increase). The 4 center paths are then combined in a single output. The design of the divider/combiner was kept simple, according to the design rules described by Saleh\textsuperscript{xii}. The measured isolation was $\sim$20 dB and the input and output reflections less than -12 dB. The operating frequency band is 2-3 GHz.

![Figure 68: Picture of calibration board.](image)

It is possible to determine phase and amplitude distributions at the output ports of such a structure by using the measured S parameters. The amplitude error is less than 5% and the phase error is less than 5° proving that this simple and inexpensive
measurement system can be used for automated diagnostics of linear and nonlinear antenna arrays.

This board allows us to calculate, at each frequency, the slope and the offset of the phase measured with respect to the actual phase. For each frequency and each of the four delays we also stored the amplitude correction factor, which takes into account the different path losses and the nonlinear amplitude and phase response of the mixers.

We implemented a Windows®-based computer program with a user-friendly interface in order to perform the automatic calibration and the actual measurement with a HP GPIB and PCI 1200 (analog-digital) network. Sample & hold devices on a printed circuit board are used to control the voltage on the varactors and thus the free running frequencies of the oscillators. Figure 66 shows a picture of the complete measurement setup.

**Measurements**

To verify the performance of the proposed calibration procedure, we measured a passive structure, such as a power divider with known delay lines. Remarkably, the phase error was less than 5° and the amplitude one less than 5%.

The real challenge is however the measurement of nonlinear active antennas for which the system has being designed. To verify the ability to perform accurate measurements, we designed and built a 5-element coupled oscillators array. We used standard surface mounted oscillators with external resonant circuitry. The
output power is ~13 dBm and the tuning range is determined by the tuning external varactor. In our design, the oscillation frequency could be tuned from 2 to 3 GHz. The locking range is measured to be 10 MHz.

The bandwidth of the patch array is somewhat limited compared to the tuning range, but as stated previously, the isolators separate the antenna behavior from the

Figure 69: Measured and expected radiation pattern for a working array.
measurement procedure. Finally, Figure 69 shows the excellent agreement between the measured and the ‘estimated’ radiation patterns of the array.

The ‘estimated’ pattern is obtained as the product of the single antenna radiation pattern and the array factor for a linear array, using the measured phase and amplitude distributions. Also in the case of defective arrays the measurement
system can be useful as shown in Figure 70. Patterns showing beams steered to several angles were obtained by applying appropriate tuning voltages to the end oscillators of the array.

**Modulation and noise**

Our system allows for steady state accurate measurement of phase and amplitude distributions. The noise performance have been already measured by past Prof. York’s students and here we just report the plot confirming the strict agreement with the theoretical results of the previous chapter.

![Graph](image_url)

Figure 71: Measured phase noise performances reported by Chang at Al". 

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187
Concerning the modulation, we were able to modulate at 1 KHz the array when all the natural frequencies are modulated together. The limited modulation range is due to the limited bandwidth of the steady state measurement system.

At JPL the modulation of the array was performed using the oscillator immediately to the right-hand side of the center oscillator and the results are shown in

![Square Wave Modulation](image)

**Figure 72: Measured and theoretical transient response of the array with the oscillator immediately to the right-hand side of the center oscillator modulated.**

As last comment, the same unit cell was then used at JPL to fabricate and test a two dimensional array which showed excellent agreement with the theoretical predictions of scanning by simple edge detuning\(^{xiv}\). The locking range measured match with design one which was based on the mutual injection of 5dBm through coupling (see Figure 52).

ii "RFIC Oscillator PM5103", datasheet from www.reel.com, for the Pacific Monolithics division.


In the previous chapters we studied the dynamical behaviors of COAs, which base their dynamics on the phenomenon of injection locking. In these last two chapters a new nonlinear phenomenon that lead coupled systems synchronize will be studied: the phase locking. Only the phase dynamics is involved in Coupled Phase-Locked Loop Arrays (CPLLAs), but the final governing equations are very similar. As we will soon see, the advantage of CPLLAs is mainly in the larger locking range and more accurate modeling which both ensure easier design and control. On the other end, the internal delay in the feedback loops becomes a critical limiting factor for nonintegrated CPLLAs. In this chapter we review the main theoretical results of the CPPLAs theory and we present some design and characterization issues.

**Single Cell Dynamics**

The standard Phase-Locked Loop circuit is shown in Figure 1. A VCO is controlled by a feedback loop that returns, after signal treatment, the instantaneous phase difference between the local oscillator output and an injection RF signal. The loop can represent pretty much any feedback linear or nonlinear block, but usually
stands for the effect of low pass filter and amplifier that remove the undesired harmonics generated by the phase detector.

![Figure 1: Basic Phase Locked Loop cell.](image)

**Dynamic Equation**

Our goal is to control the phase between the local oscillator and the injected signal and this can be achieved modifying the standard PLL by adding an offset in the loop. In order to derive a phase governing equation, we may regenerate the schematics of the system as a set of block transfer functions of the different elements composing the PLL. The best way to proceed is to first describe the linear part of the system in the Laplace domain and then combine it in the time domain with the dynamic description of the nonlinear sections.
**Voltage Controlled Oscillator**

Referring to Figure 2, the VCO by definition linearly transforms an input voltage into an output frequency or phase derivative. Thus its governing equation can be written as

\[ \dot{\theta} = \omega_c + \beta z \]  

(6.1)

where \( \beta \) is the tuning sensitivity usually measured in MHz/V in the microwave applications and \( \omega_c \) is the oscillation frequency when the input \( z \) is zero. Adding to the signal coming from the loop an external low frequency control signal \( y_{ctrl} \), then the output frequency in the absence of feedback signal becomes the equivalent of the free-running frequency of the injection locked oscillator:

\[ \omega_o = \omega_c + \beta y_{ctrl} \]  

(6.2)

In this way the detuning control becomes as simple as in the COA case.
**Phase Detector**

To detect the phase difference between local oscillator and the reference (injected) signal, several methods can be implemented, but at our knowledge the most convenient in the microwave range is the use of a double balanced mixer, which presents high isolation between the two RF inputs and is simple to use. The main drawbacks are the need for output matching not to unbalance the diode symmetry, the need for filtering higher harmonics and the conversion loss, $\gamma$, which requires the use of extra loop amplification.

Referring again to Figure 2, the DBM multiply the LO and the RF inputs into an IF output. Theoretically the IF has two contributions, one from the total phase difference and the other from their sum. When used as phase detector, the two inputs must be close in frequency and thus only their difference is desired. To eliminate the other contribution a low pass filter is necessary.

In phase terms, the DBM governing equation can be written (neglecting the higher harmonics) as

$$x = \frac{1}{2} \gamma \cos(\theta_{\text{inj}} - \theta)$$

(6.3)

where the factor of two comes from the standard trigonometric identities.

**Amplifier and Low Pass Filter**

The signal coming out from the phase detector needs to be filtered and amplified. Usually in standard applications, three types of filter are used. They are
referred based on the resulting closed loop transfer function as first order, second order with perfect integrator and second order with imperfect integrator. All of them can be represented with a generic formula:

\[
\frac{\dot{y}}{x} = \alpha \frac{1 + s\tau_z}{1 + s\tau_p} e^{-sT}
\]  

(6.4)

where we have also included the internal loop delay \(T\), as its contribution is critical in determining the stability of the loop. The tilde is associated to the Laplace transforms.

Usually PLLs are studied using linear analysis under the assumption that \(\Delta \theta\) is small. In our case the linear assumption is definitely not valid, since to understand the phase synchronization in greater detail, we have to let the phase difference approach the stability limits. For now we just assume that in the general case our circuit can be described with a pole \((\tau_p)\) and zero \((\tau_z)\). From the implementation point of view, this can be easily achieved through passive or active networks, as we will see later. The DC gain is expressed as \(\alpha\).

**Nonlinear Governing Equation**

According to the basic laws of phase locked loops, the phase of each oscillator can be changed relative to a reference input RF signal by adjusting a DC Offset added in the feedback loop. In steady state the phase difference behaves as in the phase injection phenomenon, but shifted of 90°. This occurs because the mixer DC
output has the center of its stability range when the two input signals are in quadrature. The addition of a $\pi/2$ transmission line would solve this issue.

However the phase dynamics of the unit cell presents significant differences with the injection model. In our first approach to the dynamics, we neglect the delay $T$. In this case, recalling that $s$ correspond to a derivative in the time domain, substituting (6.4) and (6.3) in (6.1) we obtain

$$\dot{\theta} + \tau_p \dot{\theta} = \omega_o + \tau_p \dot{\omega}_o + \frac{1}{2} \alpha \beta \gamma \left[ \cos(\theta_{inj} - \theta) - \tau_z \left( \dot{\theta}_{inj} - \dot{\theta} \right) \sin(\theta_{inj} - \theta) \right] \quad (6.5)$$

For sake of simplicity we consider a common reference frequency for injected and PLL frequency, for which $\theta = \omega_{ref} t + \phi$ and $\theta_{inj} = \omega_{ref} t + \psi$ as we did for the injection locking phenomenon. The reference signal can be the common oscillation frequency in the array or the injection signal or whatever frequency allows for simplification of the mathematics in the sense of removing unnecessary time dependence functions.

Using this common reference for the phasor description, we can rewrite (6.5) as function of the phases and detuning

$$\dot{\phi} + \tau_p \dot{\phi} = \omega_o - \omega_{ref} + \tau_p \left( \dot{\omega}_o - \dot{\omega}_{ref} \right) + \frac{1}{2} \alpha \beta \gamma \left[ \cos(\psi - \phi) - \tau_z (\dot{\psi} - \dot{\phi}) \sin(\psi - \phi) \right]$$

$$\quad (6.6)$$

Now it is convenient to use following definitions
\[
\Delta \phi = \psi - \phi \\
\Delta \omega = \omega_{\text{ref}} - \omega_o \\
G = \frac{1}{2} \alpha \beta \gamma
\] (6.7)

to recast (6.6)

\[
\dot{\phi} + \tau_p \ddot{\phi} = -\Delta \omega - \tau_p \Delta \dot{\omega} + G \left( \cos \Delta \phi - \tau_z \Delta \dot{\phi} \sin \Delta \phi \right)
\] (6.8)

where \( G \) represents the loop gain (usually in MHz).

Clearly (6.8) resemble the injection locked dynamics if

\[
\tau_p, \tau_z \rightarrow 0 \\
-G \cos \Delta \phi = \Delta \omega_{\text{lock}} \sin \Delta \phi
\] (6.9)

This last condition can be satisfied in several ways, for example with a 90° line and negative loop gain (the inverter configuration of an operational amplifier).

The most important results arising from (6.8) are the following:

- The locking range depends only on easily measurable and designable parameters and it’s independent of \( \omega_o \), so detuning does not affect the locking range as in the case of COAs;

- There is not amplitude dynamics in the assumption that the phase detector, amplifier and oscillator are designed correctly in order to be as close as possible to their ideal transfer functions.

In addition the locking range is determined by the loop gain, a parameter easily controlled. Finally it is known that PLLs have lower phase noise than their open
loop oscillators and the feedback loop reduces the sensibility to component tolerances.

**Steady State**

In steady state the phase difference depends on the original frequency detuning normalized to the loop gain and it’s function is shown in Figure 3

\[
\Delta \phi_{ss} = \psi - \hat{\phi} = \cos^{-1}\left( \frac{\omega_{\text{ref}} - \omega_o}{G} \right)
\]

(6.10)

![Figure 3: Steady state phase difference versus normalized detuning.](image)

Obviously when the condition (6.9) is satisfied, the curve becomes identical to the steady state of Adler’s equation. The main difference is that, since \( G \) does not
depend on the free running frequency, varying $\omega_b$ is the same as changing $\omega_{ref}$ (or $\omega_{inj}$).

**Array Dynamics**

Now that the advantage offered by the phase locking versus the injection locking is clear, we can try to exploit it in a coupled system.

**Phase Governing Equation**

All these observations led to consider PLLAs better systems to implement beam scanning with large bandwidth modulation.

To achieve nearest-neighbor coupling, a double feedback loop around the VCO must be implemented as shown in Figure 4. This coupling scheme ensures the same phase dynamics as COAs if the loops have no delay, the filters are not present and the phase detector has a sinusoidal response to phase differences. In this way a constant phase progression could be realized by adjusting the free-running frequencies of only the end elements in the array a proposed by Liao and York for COAs.

When a filter loop is taken into account higher order derivatives show up in the dynamic equation of the array.

198
Assuming identical unit cells and neglecting again the loop delay, we can express the phase dynamics as function of the respective detuning and phase difference between neighbor cells, by defining

\[
\Delta \phi_i = \phi_i - \phi_j \\
\Delta \omega_i = \omega_i - \omega_{ref}
\]  

(6.11)

Neglecting the effect of injection at the edge elements (as it would be in a Stephan’s approach), with a procedure analog to the one for the single cell and using the previous definition, we can arrive to the governing equation of the PLLAs:

\[
\dot{\phi}_i + \tau_p \ddot{\phi}_i = -\tau_p \Delta \dot{\omega}_i - \Delta \omega_i + G \sum_{j=1\neq i}^{j=i+1} \left(-\tau_z \Delta \dot{\phi}_j \sin \Delta \phi_j + \cos \Delta \phi_j \right)
\]  

(6.12)

When (6.9) are verified it is clear the resemblance with the CO phase dynamics:

\[
\dot{\phi}_i = -\Delta \omega_i - \Delta \omega_{lock} \sum_{j=1\neq i}^{j=i+1} \sin \left(\phi_i - \phi_j\right)
\]  

(6.13)

In steady state the equations become

\[
\Delta \omega_i = -\Delta \omega_{lock} \left(\sin \Delta \phi_{i,i+1} + \sin \Delta \phi_{i,i-1}\right) \quad \text{COA}
\]

\[
\Delta \omega_i = G \left(\cos \Delta \phi_{i,i+1} + \cos \Delta \phi_{i,i-1}\right) \quad \text{CPLLA}
\]  

(6.14)

Thus we are expecting that the two arrays will have similar behaviors, with the CPLLA being more performance and easy to model. Thus also the beam scanning
by edge detuning may be implemented as it is based purely on the steady state equations (6.14).

**Matlab Implementation**

As we did for CO, an accurate model of CPLLAs can be implemented in Matlab, since all the constitutive block can be represented in Simulink. Also the effect of delay and non-identical cell can be simulated. The model is composed by groups of two subsystems, one representing the VCO and the other the feedback loop as shown in Figure 4, Figure 5 and Figure 6.

**Figure 4:** Matlab implementation of the feedback loop for CPLLAs.

**Figure 5:** Matlab implementation of the VCO for CPLLAs.
Figure 6: Simulink implementation of a 5 elements CPLLA.

Another advantages of such simulation is that after obtaining from the characterization of a real single cell, its parameters can be input in the model to obtain an idea on what the performance of an array would look like for different values of $N$, coupling network, and other possible variable in the system.

As we will soon see, the beam scanning by edge detuning and several modulation schemes could be evaluated thanks to this discrete modeling.
Also the noise analysis may be performed following the guideline mentioned in the first chapter and Appendix A.

Thus even if the model of PLLAs is a little bit more complex at first sight than COAs, its accuracy justify the effort….

Continuum Modeling

It is natural to think to extend the continuum analysis of COA to PLLAs. In order to simplify our approach, we assume to have identical PLLs and coupling parameters, to neglect $\tau_z$ and injection sources. We also assume to have the 90° lines and the negative loop to be able to compare the results with the CO ones.

When the number of PLLs increases and the distance reduces, following the same procedure used in the derivation of the continuum equation for COA, then $\sin \Delta \phi \sim \Delta \phi$ and from (6.12) we obtain the continuum phase dynamic equation

$$\frac{\partial^2 \phi}{\partial \chi^2} - \frac{\partial \phi}{\partial \tau} - r \frac{\partial^2 \phi}{\partial \tau^2} = - \frac{\omega_{\text{tune}} - \omega_{\text{ref}}}{G} - r \frac{\partial \omega_{\text{tune}}}{\partial \tau}$$  \hspace{1cm} (6.15)

where we defined

$$\tau = Gt$$

$$r = G \tau_p$$ \hspace{1cm} (6.16)

While the first is just the normalization of the time to the hold-in (or locking) range, the second term is the coefficient of the second derivative in time which
does not appear in the phase dynamics of COAs. Based on this parameter, the
dynamic equation goes from a heat flow to an electrodynamics type of behavior.

![Diagram](image)

**Figure 7: CPLLA response to edge detuning: the system goes to a linear phase distribution.**

A typical response to edge detuning is shown in Figure 7, confirming the ability
of CPLLAs of easy shifterless beam scanning. At first look the response seems to
be steeper than COAs. To verify such prediction, we plotted in Figure 8 the
evolution of the output phase of the edge element as response to a step detuning
used for beam steering. Clearly when $r = 0$, we obtain the same equation of COAs.
Increasing $r$ the response is a little bit faster.
However a more significant quantity is the phase difference between two neighbors, recalling that only the PLL located in discrete points have a physical meaning. The result is plotted in Figure 9, where clearly the steady state is reached much faster as $r$ increases. Also note that while the desired value of a particular phase is achieved on a time scale of hundreds of locking ranges, the phase difference steady state is reached within tens of the hold-in range.
Locking Diffusion

The discrete model simulated in Matlab confirms the continuum model (heat transfer). First the diffusion of the locking progresses away from the detuning or injection source. So if the detuning occurs at the edges, the center elements will be the latest to reach their steady state values as shown in Figure 10.

Figure 10: Progressive locking from the edges (detuning) to the center in a seven-element array.

Another heat diffusion type effect is related with the array size. When this increases it takes longer time for the center point to reach its steady state. The same situation was occurring with COAs, as the transient response was related to the
square of the array size. This phenomenon is shown in Figure 11 for several number of elements.

![Figure 11: Center element phase locking as the array size increases.](image)

**Coupling and Loop Gain**

Concerning the coupling network for CPLLLAs, the nearest-neighbor coupling scheme proposed above present the advantage of being simple to implement and showing the promise of easy beam steering. Nevertheless, other two issues have been addressed.

First, as for COAs, the coupling phase plays an important role together with the loop gain sign in determining where the 180° phase difference range will be
centered. It can be shown that to obtain a broadside beam, the PLLs must have negative loops and $\pi/2$ coupling lines or positive loops and $3\pi/2$ coupling lines. Other configurations will create endfire beams.

As it was shown for COAs, the phase of the coupling line determines the stability region for the array behavior. Also for CPLLAs we found a similar situation (Table 1), but this time instead of the oscillator modeling, the cell parameter which link to the coupling phase is the sign of the loop gain as it can be argued directly from (6.9).

<table>
<thead>
<tr>
<th>Beam Type</th>
<th>Loop Gain Sign</th>
<th>RF Coupling Line Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Endfire Neg.</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td>Endfire Pos.</td>
<td>−</td>
<td>0</td>
</tr>
<tr>
<td>Broadside</td>
<td>−</td>
<td>$\pi/2$</td>
</tr>
<tr>
<td>Endfire Neg.</td>
<td>−</td>
<td>$\pi$</td>
</tr>
<tr>
<td>Endfire Pos.</td>
<td>+</td>
<td>$\pi$</td>
</tr>
<tr>
<td>Broadside</td>
<td>+</td>
<td>$3\pi/2$</td>
</tr>
</tbody>
</table>

*Table 1: Endfire or broadside beam versus coupling phase and loop gain sign.*

Simulations in Matlab confirmed our predictions as shown in Figure 12.
Second, as intuition suggests longer the line, longer the delay associated with the phase information and thus slower the locking along the array. The CPLLAs with the delay line are intrinsically asymmetric, and thus the longer the coupling line, the longer and more asymmetric will be the locking phenomenon as shown in Figure 13. Thus even if theoretically multiple of $2\pi$ in line length should not affect stability, longer lines have larger offsets and transients.
Single Cell Design and Characterization

As we previously stated, adding an offset in the loop to control the phase between the local oscillator and the injected signal modifies the standard PLL design. The basic unit cell is presented in Figure 14. A three way summer allows for both the mutual coupling as well as the phase control $V_{ctrl}$ mentioned before (adjustable center frequency).
Figure 14: Schematics of the CPLLAs unit cell.

A PLL to operate at 2.5 GHz was designed in Ads, build and characterized using standard measurement techniques implemented in Labview. We measured large locking range up to 250 MHz as the gain increases. The amplitude fluctuations were very small (1 dB). The circuit was reliable and insensitive to component tolerances. The picture of the text fixture for our PLL is in Figure 15.

Figure 15: Picture of the CPLLAs unit cell.
**PLL Characterization**

To characterize the lines for the correct phase locking, we built the test fixture in , where a in injection signal is divided in two and goes from one side to the input of the MTA and the other to inject the PLL. In this way we can compare accurately the synchronization and the phase difference between the two signals.

![Test fixture for the PLL characterization.](image)

*Figure 16: Test fixture for the PLL characterization.*

Changing the gain we can reach a hold-in range (2 $G$) of 250 MHz as shown in Figure 17. The capture (or pull-in) range is 5,1 MHz. A clear difference between the simple injection (open loop) locking range and the phase locked (close loop) gain is highlighted.
The phase difference between injection signal and local oscillator within the hold-in range was measured using a microwave transition analyzer and it’s shorn in Figure 18.

Similar behavior is observed for mutually coupled PLLs. A 1.5 GHz was build and measured\textsuperscript{iii} and it is shown in Figure 19.
Phase Detector

As we stated before, a good microwave phase detector is a double balanced mixer, since it has high LO/RF isolation and its cheap and simple to use. However its design cannot be isolated from the following loop block, since there is a need to filter harmonics as shown in Figure 20.

Figure 19: Two mutually coupled PLL at 1.5 GHz and the locking range characterization.
Moreover high LO level and 50-Ω load are required. Finally it can be unbalanced by DC offsets. A good solution to these issues is shown Figure 21.

An RF amplifier boost the VCO power into the DBM and the IF filtering is achieved through a 50 W passive network which also reduces the effect of the DC offsets of the operational amplifiers. The resulting IF is plotted in Figure 22.


Internal Delay

The main drawback of PLL makes with off the shelf components is that the feedback loop may have a considerable amount of transport delay compared to the free running frequency. In COAs this delay was intrinsically small since no external loop was present. Obviously the delay is drastically reduced in fully integrated circuitry such MMICs. Accurate Matlab modeling shows the effects of delay in the feedback loop while setting design tradeoffs. Experimentally was observed the presence of undesired periodic solutions for high gains. Recent advances in understanding the mathematics behind such phenomenon, lead to the definition of new design tradeoffs.

Understanding the behavior of two coupled loops helps us understand how to build larger arrays and to understand the effect of the internal delay. In the case of two coupled PLLs the phase equation becomes (see also Figure 19):

$$\tau_p \ddot{\phi} + (1 + \tau_z G \sin \Delta \phi) \Delta \dot{\phi} - G \cos \Delta \phi = \Delta \omega$$

(6.17)
where $\Delta \phi$ is the phase difference, $\Delta \omega$ is the frequency detuning, $G$ is the loop gain and $\tau_p$ and $\tau_z$ are the filter zero and pole constants.

From (6.17) two quantities can be defined. Within the hold in range, $\Omega_h$, the oscillators remain locked. In the pull-in range, $\Omega_p$, the oscillators will come to lock. These can be evaluated as:

$$\Omega_h = 2G$$

$$\Omega_p = 2 \sqrt{\frac{1 + 4\tau_p^2 G^2 - 1}{2\tau_p^2}}$$  \hspace{1cm} (6.18)

Buckwalter and York shown that from the characteristic constants in the solution of (6.17) as a function of $G$, $\tau_p$ and $\tau_z$ presents a bifurcation. The presence of a pole causes the solution to bifurcate for a particular gain, below which the acquisition time, determined by the slowest constant, diminishes. Above that value if the zero is taken into account, the gain increase improves the acquisition time.

In real systems increasing the gain brings the system to unlock. To be able to account for this phenomenon, a delay must be introduced in the feedback loop.

This can be modeled enough accurately with the Pade’s approximation

$$e^{-st} \approx \frac{1 - \frac{1}{2}sT}{1 + \frac{1}{2}sT}$$ \hspace{1cm} (6.19)
In terms of linear theory, this approximation introduces a RHP zero which will make the feedback system potentially unstable, since the locked state does no longer satisfy the condition of unconditional stability.

![Figure 23: Optimal gain (blue) and critical gain (red) limit the value of the loop gain, when introducing a delay and a filter in the PLL feedback.](image)

The solution of (6.17) still presents the bifurcation as before, but the gain increase after the bifurcation causes also an increase of the acquisition time (Figure 23). Further increase of the gain lead to unstable negative solutions. Thus we can now define a range for the loop gain from the optimal gain to the critical gain. This range is strongly dependent from the delay value.
In summary: above the critical gain the loop locks in limit cycle (undesired sideband modulation as periodic solutions) and below the optimal gain, for which we obtain the smallest time constant (fastest lock) there is bifurcation of the solutions.

![Figure 24: Experimental verification of the instability introduced by the loop transport delay.](image)

Experimental data fully confirms the delay theory as shown in Figure 24.

In integrated PLL, the effect of this delay is negligible. On the other hand, in discrete assembled PLL (as the one used in CPLLAs’ prototypes) it limits the max gain loop and thus the locking range.
Array Design and Characterization

A five-element 2.45 GHz coupled PLL array was build and tested (Figure 25).

Figure 25: A five-element 2.45 GHz coupled PLL array.

Based on the previous theory, we design the array for broadside radiation, thus the loop gain was negative and the coupling line phase shift was $\pi/2$. The other important parameters are:

$$G = 200 \text{ MHz} \quad \tau_p = 40 \text{ MHz} \quad T \approx 1 \text{ ns}$$

As we will see more in details later, the total phase difference ranges from $-315^\circ$ to $+300^\circ$. Digital frequency modulation by global control of the free running frequencies can be done up to 10 MHz when the beam is centered.

We verified immediately the ability to lock with a simple setup as shown in Figure 26, where the free running frequencies are slowly changed to reach the
capture range of the center element leading the array to a progressive synchronization.

Figure 26: Experimental verification of the synchronization process.

**Beam Scanning**

![Beam Scanning Diagram]

Figure 27: Schematics of the simple shifterless beam scanning by edge detuning.
The beam scanning ability by edge detuning (Figure 27) has being experimentally verified, as shown in here the radiation beam is steered of 15°, as shown in Figure 28.

![Figure 28: Beam scanning by edge detuning of a 5-element 2.45 GHz CPLLA.](image)

Also the transient associated with the edge-detuning can me measured using a fast oscilloscope with multiple input ports. The interesting feature is that the offset due to the array coupling asymmetry is present as shown in Figure 29. Moreover the heat-type dynamics is evident as the elements at the sides reach steady state faster than the inner elements.
When also the modulation needs to be applied, the same discussion about modulation schemes valid for COAs can be applied to CPLLAs. The main difference is that here the modulation is less sensitive to $\omega_0$, since the all the dynamic parameters are independent of it. A typical implementation is shown in Figure 30, where all the low frequency $y_{\text{ctrl}}$ are modulated simultaneously (global control). We should first of all note that to implement a linear span of the modulation, appropriate nonlinear control stages should be used in the tuning lines as the phase and free running frequency differences have a sinusoidal dependence.
Since there is no amplitude dynamics, Am can be performed only through additional voltage controlled amplifiers or attenuators. Even though this solution is simple, it can be become complex from a layout point of view as the number of control lines would increase with the array size.

Frequency modulation can be easily achieved without affecting the scan simply by globally modulating $\omega_0$. The beam fluctuations are caused by the fact that the coupling lines have a different phase at different frequencies. However this effect is relatively negligible and occurs also in standard phased arrays.
On the other side, since the detuning and the modulation contribute to the value of the feedback voltage, there must be a design tradeoff between desired maximum beam angle and maximum modulation span.

$$V_{\text{det}} + V_{\text{mod}} = V_{\text{ctrl}}$$  \hspace{1cm} (6.21)

The FM scheme was implemented in Matlab assuming identical PLL and the results of the simulations are shown in Figure 32. It is clear that the beam scanning is preserved during modulation.
Figure 32: Matlab FM implementation with identical PLLs.
When measured with the fast oscilloscope the CPLLA shows agreement with the theoretical predictions. Comparing the transient time with the one of COA in the previous chapter, it is clear the improvement of the modulation response of the system. These results demonstrate that PLLAs are a viable solution for shifterless beam scanning and frequency modulation. When an external reference injection signal is used, also phase modulation can be achieved by global control of the free running frequencies as the whole array locks in frequency to the injection and thus behaves closely to a single PLL.

Figure 33: Transient of the frequency modulation by global control of the 5-element CPLLA.
Comments

In conclusion the recent studies on CPLLAs have shown their limitations. The corrections applied to the models improved our understanding of these systems that showed to be more reliable and predictable than COAs.

We want to point out that a new idea that combines the improved locking range of PLL with the noise performances of the injection-locked oscillator was proposed by ?v,vi. The subharmonic injection locking phase locked loops enhances drastically the operating frequency, the locking range and the phase noise. This circuit could be also embedded in a coupling network with the potential of improving the overall performance of the array.

Also the scanning range could be improved by introducing frequency divider before the phase detector. This would be similar to the frequency doubler circuit after each oscillator suggested by Alexanian et Alvii, which effectively doubles the inter-element phase shift.

Here we only presented the transmitting operation of CPLLAs. This is actually not a limitation of such systems. The edge-detuned scanning configuration can also be employed in a receiving application in a configuration similar to the one suggested for COAsviii. This is accomplished by using the scanning oscillator array as the local oscillator for a set of mixers. It may be possible to merge the transmit and receive functions, especially for FMCW imaging arrays, by making each array element a self-contained FMCW transmitter and receiver, with each array element
coupled to its neighbors. Alternatively, each array element could be a self-oscillating mixer. These concepts have not yet been tested.

---


We presented the recent findings in injection locking and phase-locked loop coupled arrays. While the COAs have been extensively studied in the last two decades, the CPLLAs have been only recently proposed to overcome the limitations of COAs. During the research activity presented in this dissertation we improved the understanding of synchronized nonlinear phased arrays while verifying experimentally some of the features that makes such systems promising in the communication market and radar industry.

Coupled Oscillator Arrays base their dynamics on the injection-locking phenomenon. The governing equations of ideal COAs showed the ability of controlling a linear phase distribution by edge detuning as well as reducing the overall phase noise. Also a full range of nonlinear behaviors is observed experimentally and can be derived from the dynamic equations.

The discrete theory used in the past to describe the behavior of such systems even if relatively accurate lacks of easy understanding. We contributed to the development of a novel theoretical approach, based on a continuum modeling of COAs. It allows for a more intuitive analysis of the dynamics, based on heat
diffusion and electrostatic analogies. With this approach: two important properties of the system are verified: beam scanning by edge detuning and modulation by global control of the free-running frequencies.

We build a five-element COA to verify the theoretical predictions a 2.5 GHz. A system capable of measuring the steady state output phase and amplitude distributions was needed. Good design of such system lead to very good agreement between the expected and measured beam steering results. In collaboration with JPL a seven-element array was also build and tested, showing the ability to be phase modulated. JPL extended the results and experiments to a nine elements 2D array.

The main drawbacks of such systems are the dependence of the locking range to the free running frequency and the coupling between amplitude and phase dynamics. If the synchronization is achieved through the phenomenon of phase lock this problems can be solved. This motivated our efforts toward Coupled Phase-Locked Loop Arrays. CPLLAs are still in their infancy, but recent studies showed their limitations as well as their potential for reliable low-cost beam scanning systems. The recently modified models of CPLLAs offer the tools for more predictable and performing systems. We developed a continuum model for such systems and implemented the discrete analysis in Matlab. Both showed a ‘heat’-like behavior similar to the one of COAs but the transient performances are improved. Moreover, since standard circuit blocks can accurately describe the
system, the theoretical predictions agree strictly with measurement data. The main issue arising from both theoretical studies and experiments is related to the effects of the loop and coupling delays. The coupling delay determines together with the loop gain sign the stability region and thus which type of radiation pattern is expected, broadside or endfire. On the other end the loop delay limits the gain confining it below a critical value and above a bifurcation point. While this delay effects drastically design using discrete components, it can be easily solved by device integration.

Future research activity may focus on finding analytical solutions for continuum model of CPLLAs and extend it to include injection and the filter zero. Also the experimental verification of: Stephan’s scanning and modulation schemes could be of interest as well as the noise analysis and its Matlab implementation.

Being governed by strongly nonlinear behaviors, still a lot needs to be understood about these synchronized arrays: the identification of other attractive features and limitations, will be particularly useful in future communications.
Consider a linear, time-invariant network described by the voltage transfer function $H(\omega)$,

$$\tilde{V}_{out}(\omega) = H(\omega)\tilde{V}_{out}(\omega) \quad (A.1)$$

This equation can be expressed in the time domain using the Fourier transform pair

$$\tilde{V}(\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t} \, dt \leftrightarrow v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}(\omega)e^{j\omega t} \, d\omega \quad (A.2)$$

If the input signal $V_{in}$ is a modulated sinusoid at a carrier frequency $\omega_0$, we can represent the time variation in the complex form

$$v_{in}(t) = A(t)e^{j\left[\omega_0 t + \phi(t)\right]} = v_{in}'(t)e^{j\omega_0 t} \quad (A.3)$$

where it is understood that the actual time variation is found by taking the real part of (A.3). Note that the spectrum of the modulation alone,

$$\tilde{V}_{in}'(\omega) = \int_{-\infty}^{\infty} v_{in}'(t)e^{j\omega t}e^{-j\omega_0 t} \, dt \quad (A.4)$$

is related to the input signal spectrum by the well-known shifting property.
\[ \tilde{V}_i''(\omega) = \tilde{V}_i''(\omega + \omega_0) \] (A.5)

Taking the inverse Fourier transform of (A.1) gives

\[ v_{out}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \tilde{V}_i''(\omega) e^{j\omega t} d\omega \] (A.6)

\[ v_{out}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega' + \omega_0) \tilde{V}_i''(\omega') e^{j(\omega' + \omega_0)t} d\omega \] (A.7)

The transfer function can be expressed as a Taylor series about the carrier frequency \( \omega_0 \),

\[ H(\omega' + \omega_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n H(\omega_0)}{d\omega^n} (\omega')^n \] (A.8)

Substituting (A.8) into (A.7) and interchanging the order of integration and summation gives

\[ \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n H(\omega_0)}{d(j\omega)^n} (\omega')^n \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega)^n \tilde{V}_i''(\omega') e^{j\omega't} d\omega \right] e^{j\omega_0 t} \] (A.9)

The integral in square brackets is recognized as the nth derivative of the modulation, so from (A.3) we can write

\[ v_{out}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n H(\omega_0)}{d(j\omega)^n} \frac{d^n}{dt^n} (Ae^{j\phi}) e^{j\omega_0 t} \] (A.10)

The actual output is found by taking the real part of (A.10). In the case of a slowly varying modulation and/or a broadband network, the higher-order
derivatives in (A.10) diminish rapidly and the output voltage is well described by the first two terms of the expansion. Taking the first two terms and using (A.3), (A.10) reduces to

\[ v_{\text{out}}(t) \approx v_{\text{in}}(t) \left[ H(\omega_0) + \frac{dH(\omega_0)}{d\omega} \left( \frac{d\phi}{dt} - j \frac{1}{A} \frac{dA}{dt} \right) \right] \]  

(A.11)

Kurokawai appears to be the first to recognize that this result can be obtained from the state equation (A.1) by substituting

\[ \omega \rightarrow \omega_0 + \frac{d\phi}{dt} - j \frac{1}{A} \frac{dA}{dt} \]  

(A.12)

in the transfer function, and invoking a “slowly-varying amplitude and phase” assumption,

\[ \frac{d\phi}{dt} \ll \omega_0 \quad \text{and} \quad \frac{1}{A} \frac{dA}{dt} \ll \omega_0 \]  

(A.13)

to expand the transfer function around the frequency \( \omega_0 \). This is referred to as the “Kurokawa substitution” in the text.

Note also that we did not have to choose the carrier frequency \( \omega_0 \) as the reference point for the Taylor expansion in (A.8). It is a natural choice in this example, but in the case of a multiple oscillator system, there are several “natural” frequencies to choose from. We can express the input signal (A.3) in terms of any arbitrary reference frequency by suitably redefining the phase variable to include a linearly increasing part,
\[ v_{in}(t) = A(t)e^{i(\omega_0 t + \phi(t))} \]

(A.14)

where \( \phi'(t) = \phi(t) + (\omega_0 - \omega_r)t \). The resulting expression for \( v_{out}(t) \) is then exactly the same as (A.3) if \( \omega_r \) replaces \( \omega_0 \), and \( \phi' \) replaces \( \phi \). One must be careful, however, when truncating the expansion under the assumption of slowly varying amplitude and phase in this case. In fact \( \phi' \) can vary rapidly in time compared with \( \phi \) if the new reference frequency \( \omega_r \) differs significantly from the "true" carrier frequency, which was assumed to be \( \omega_0 \) in the example.

The following are explanations of some of the results reported in the various chapters and for which the detailed mathematical proof would have diverted the attention of the reader from the important results.

**B1: Some FFT Basics for Spectral Power Analysis**

The goal of spectral estimation is to describe the distribution (over frequency) of the power contained in a signal, based on a finite set of data. Estimation of power spectra is useful in a variety of applications, including the detection of signals buried in wide-band noise.

The correlation function of a stationary random process \( x(t) \) is mathematically defined as

\[
r_{xx}(\tau) = \langle x(t) x(t+\tau) \rangle_t = \int_{-\infty}^{\infty} x(t) x(t+\tau) dt
\]

(B.1)

The total signal power can be evaluated in \( \tau = 0 \). The power spectrum of such process is related to the Fourier transform of the correlation function.
The interesting quantity is however not the power spectrum as in the case of
deterministic signals, but the power spectral density (PSD). PSD is a quantity easier
to measure practically, since band limited instrumentation allows the measurement
of the power in a particular frequency range. Moreover the background noise and
the one intrinsic to the measurement system (usually modeled as white noise) have
limited PSD, but colossal power if integrated in all frequencies and would cover the
interesting signals. PSD is thus defined as the density of the power with respect to
frequency:

\[ P_{xx}(f) = \frac{dR_{xx}(f)}{df} \]  \hspace{1cm} (B.3)

These formulas can be transformed into discrete quantities to be able to apply
numerical techniques\(^{ii}\). Usually the generation of a single random process will lead
to errors due to the limitation of the time window and sampling resolution. To
 correct for these errors, several sequences must be taken in account and averaged in
the frequency domain

\[ \left\langle r_{xx}(\tau) \right\rangle_{\text{samples}} = R_{xx}(\tau) \underbrace{\text{Fourier Linearity}}_{\text{\longrightarrow}} \left| X(f) \right|^2 = \left\langle \left| X(f) \right|^2 \right\rangle_{\text{samples}} \]  \hspace{1cm} (B.4)

In our simulations we used fifty random sources, which simultaneously
generate uncorrelated signals. These signals are send one by one to the system
under investigation (i.e. the oscillator, the PLL or the arrays); the several outputs are then transformed in the Fourier domain through FFT and their power spectral densities are averaged.

Several time windows can be chosen to reduce the sidelobes caused by the limited time window, but we decided to stick with the rectangular shape, as the other would affect the resolution as shown in Figure 106. From the figure is also clear that the number of samples is a tradeoff between calculation time and accuracy of the PSD. Since we focus our attention on the noise between 1 KHz and 10 MHz from the carrier, we found a good compromise using the settings below:

\[
\begin{align*}
\text{Maximum Frequency} & = F_{\text{max}} = 10 \text{ MHz} \\
\text{Minimum Frequency} & = F_{\text{min}} = 1 \text{ KHz} \\
\text{Number of FFT Points} & = N_p = 2^{15} \\
\text{Sample Time} & = T_s = 50 \text{ ns} \\
\text{Number of Averaged Samples} & = N_s = 50 \\
\text{Noise Total Power} & = P_l = 1 \text{ W in 10 MHz band}
\end{align*}
\]

It is important to note that the FFT is calculated with equal interval in time and frequency. Thus a large amount of point will be confined in the higher spectrum when the frequency is in logarithmic scale. To overcome this issue we collect (using Excel) the FFT points in a nonlinear fashion keeping the distribution of samples constant per decade.
To evaluate correctly the SPD of a signal, a coherent choice of normalization needs to be done. We decided to use the single side normalization. This means that we assume that all the power can be obtained integrating the PSD on one side of the spectrum. Thus a white noise with power of 1 W distributed in a 10 MHz band will have a uniform spectral power density of –70 dB. When this power is integrated, its power density will be modified by the transfer function of the integrator:

\[ P_{\text{IntWhite}}(f) = \left| H(f) \right|^2 P_{\text{White}}(f) = \frac{P_{\text{White}}(f)}{(2\pi f)^2} \]  

(B.6)

and in dB, the PSD will follow a 20 dB/decade slope

\[ \left[ P_{\text{IntWhite}} \right]_{\text{dB}} = \left[ P_{\text{White}} \right]_{\text{dB}} - 20\log_{10}(2\pi f) \]  

(B.7)

The Matlab simulation of white noise and its 1/f² sloped integral are shown in Figure 107. The two curves intersect in \( f = f_c = 1/2\pi \). In a general case, the
crossing point will be determined by a characteristic integration factor, for which the $1/f$ noise can be represented as the integration of a noise floor. Usually $P^{\text{Floor}}_{xx}$ and $f_c$ are experimentally determined.

$$
\left[ P_{xx}^{1/f} \right]_{dB} = \left[ P^{\text{Floor}}_{xx} \right]_{dB} - 20\log_{10}\left( \frac{f}{f_c} \right)
$$

(B.8)

![Figure 107: Comparison between theoretical and simulated PSD of a 10^-7 W/Hz white noise and its integration using the values in (B.5).](image)

A last observation: the simulated integration of the white noise presents a small curvature in the PSD around the maximum frequency. This is clearly an aliasing effect for which the other side of the spectrum is repeated around half of the sampling frequency and consequently power summation occurs. In our case the effect is drastically reduced since the integration transfer function decreases very fast. It does not occur in the flat PSD case, because in Matlab the white noise is band limited to half of the sample frequency.
**B2: Noise of VDPO Injection-Locked to a Noiseless Source**

Our final goal is to separate the contributions of phase and amplitude fluctuations to the PM and AM noise, starting from the coupled noise equations of the injection locked VDPO

\[
\tilde{\delta} A = -\frac{\hat{A}\hat{G}_n + \rho \sin(\psi - \hat{\phi})}{j\frac{\omega}{\omega_{3dB}} - \mu\left(1 - 3\frac{\hat{A}^2}{\alpha^2}\right)}\tilde{\delta} \phi \tag{B.9}
\]

\[
\tilde{\delta} \phi = -\frac{\tilde{B}_n + \frac{\rho}{A^2} \sin(\psi - \hat{\phi})}{j\frac{\omega}{\omega_{3dB}} + \frac{\rho}{A} \cos(\psi - \hat{\phi})} \tilde{\delta} A \tag{B.10}
\]

We can cast the previous equations in a matrix form

\[
\begin{bmatrix}
j\frac{\omega}{\omega_{3dB}} - \mu\left(1 - 3\frac{\hat{A}^2}{\alpha^2}\right)
\end{bmatrix}
\begin{bmatrix}
\tilde{\delta} A
\end{bmatrix}
+ 
\begin{bmatrix}
\rho \sin(\psi - \hat{\phi}) \\
\frac{\rho}{A^2} \sin(\psi - \hat{\phi})
\end{bmatrix}
\begin{bmatrix}
\tilde{\delta} \phi
\end{bmatrix} = \begin{bmatrix}
-\hat{A}\hat{G}_n \\
-\tilde{B}_n
\end{bmatrix} \tag{B.11}
\]

\[
\begin{bmatrix}
\rho \sin(\psi - \hat{\phi}) \\
\frac{\rho}{A^2} \sin(\psi - \hat{\phi})
\end{bmatrix}
\begin{bmatrix}
\tilde{\delta} A
\end{bmatrix}
+ 
\begin{bmatrix}
\rho \cos(\psi - \hat{\phi}) \\
\frac{\rho}{A} \cos(\psi - \hat{\phi})
\end{bmatrix}
\begin{bmatrix}
\tilde{\delta} \phi
\end{bmatrix} = \begin{bmatrix}
-\hat{G}_n \\
\tilde{B}_n
\end{bmatrix} \tag{B.12}
\]

Inverting the matrix and rearranging we obtain

\[
\begin{bmatrix}
\tilde{\delta} A
\end{bmatrix} = 
\begin{bmatrix}
-\frac{\rho}{A^2} \sin(\psi - \hat{\phi}) \\
-\frac{\rho}{A} \cos(\psi - \hat{\phi})
\end{bmatrix}
\begin{bmatrix}
\tilde{B}_n \\
\tilde{G}_n
\end{bmatrix}
+ 
\begin{bmatrix}
\rho \sin(\psi - \hat{\phi}) \\
\frac{\rho}{A^2} \sin(\psi - \hat{\phi})
\end{bmatrix}
\begin{bmatrix}
\tilde{\delta} \phi
\end{bmatrix} \tag{B.13}
\]
\[
\tilde{\phi} = \frac{\rho \sin(\psi - \phi) \hat{A} \hat{G}_n \left[ j \frac{\omega}{\omega_{3dB}} - \mu \left( 1 - 3 \frac{\hat{A}^2}{\alpha^2} \right) \right] \hat{B}_n}{\left[ j \frac{\omega}{\omega_{3dB}} - \mu \left( 1 - 3 \frac{\hat{A}^2}{\alpha^2} \right) \right] \left[ j \frac{\omega}{\omega_{3dB}} + \frac{\rho}{A} \cos(\psi - \phi) \right] - \frac{\rho^2}{A^2} \sin^2(\psi - \phi)}
\]

(B.14)

The output phase and noise fluctuations are now explicitly dependent on the input noise contributions. To be able to neglect the AM output noise, since the imaginary parts are the same, we need to impose the condition (assuming \( \langle \hat{B}_n \cdot \hat{G}_n^* \rangle = 0 \))

\[
\left| \frac{\rho \sin(\psi - \phi)}{A} \hat{B}_n \right|^2 + \left| \rho \cos(\psi - \phi) \hat{G}_n \right|^2 \ll \left| \rho \sin(\psi - \phi) \hat{A} \hat{G}_n \right|^2 \left| \hat{B}_n \mu \left( 1 - 3 \frac{\hat{A}^2}{\alpha^2} \right) \right|^2
\]

(B.15)

Since \( |\hat{B}_n| \approx |\hat{G}_n| \) and \( \left| \mu \left( 1 - 3 \frac{\hat{A}^2}{\alpha^2} \right) \right| \leq 2|\mu| \), an important result arises: if the injection strength is small compared to the nonlinear parameter, \( \rho \ll \mu \), we can assume that only \( |\tilde{\phi}| \) is relevant to the noise. In such case the PSD becomes

\[
|\tilde{\phi}|^2 = \frac{\left| \hat{B}_n \right|^2}{\left\{ \frac{\omega^2}{\omega_{3dB}^2} + \frac{\rho^2}{A^2} \cos^2(\psi - \phi) \right\}}
\]

(B.16)
**B3: Noise of VDPO Injection-Locked to a Low-Noise Source**

We assume that only the phase dynamics is relevant for the noise behavior. In such case, when the reference source has low phase noise, we can again apply the perturbation analysis, \( \hat{\phi} + \delta \phi \) and \( \hat{\psi} + \delta \psi \), starting from the phase equation

\[
\frac{d \phi}{dt} = \omega_o - \omega_{inj} + \omega_{3\text{dB}} \frac{P}{A} \sin(\psi - \phi) - \omega_{3\text{dB}} B_n(t) = f(\phi, \psi) \quad (B.17)
\]

\[
\frac{d \left( \hat{\phi} + \delta \phi \right)}{dt} = \frac{d \left( \delta \phi \right)}{dt} \approx \underbrace{\frac{\partial \dot{f}}{\partial \psi} (\hat{\psi}, \hat{\phi}) \delta \psi + \frac{\partial \dot{f}}{\partial \phi} (\hat{\psi}, \hat{\phi}) \delta \phi}_{SS \rightarrow \omega_o, B_n(t)}
\]

\[-\omega_{3\text{dB}} B_n(t) + \omega_{3\text{dB}} \frac{P}{A} \cos(\hat{\psi} - \hat{\phi}) \delta \psi - \omega_{3\text{dB}} \frac{P}{A} \cos(\hat{\psi} - \hat{\phi}) \delta \phi \quad (B.18)^{\text{ii}}
\]

Transformation in the Fourier domain and averaging on ensemble lead to the calculation of the phase noise PSD (assuming \( \langle \hat{B}_n \cdot \hat{\delta \phi}^* \rangle = 0 \))

\[
|\hat{\delta \phi}|^2 = \left| \hat{B}_n \right|^2 + \left( \frac{P^2}{A^2} \cos^2(\hat{\psi} - \hat{\phi}) \right) |\hat{\delta \psi}|^2 \quad (B.19)
\]

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