Uniform Plane Waves

The Wave Equation in Lossless, Source-free Regions:

Maxwell's two curl equations can be combined to give the two vector wave equations:

$$\nabla^2\overline{E} - \mu\epsilon\frac{\partial^2\overline{E}}{\partial t^2} = 0 \qquad \nabla^2\overline{H} - \mu\epsilon\frac{\partial^2\overline{H}}{\partial t^2} = 0$$

A general solution to the wave equation consists of travelling waves which move with velocity

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

For time-harmonic, or sinusoidal signals, the wave equation becomes the Helmholtz equation

$$\nabla^2\overline{E} + k^2\overline{E} = 0 \qquad \text{and} \qquad \nabla^2\overline{H} + k^2\overline{H} = 0$$

where the wave number, or propagation constant, k, is given by

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{v} = \frac{2\pi}{\lambda}$$

Uniform Plane Waves:

A general solution to the wave equation is found using the Method of Separation of Variables, giving

$$\overline{E} = \overline{E}_0^+ e^{-\jmath\overline{k}\cdot\overline{r}} + \overline{E}_0^- e^{+\jmath\overline{k}\cdot\overline{r}}$$

where the $\overline{k}$ vector points in the direction of propagation and is defined such that $|\overline{k}| = \omega\sqrt{\mu\epsilon}$, and $\overline{r}$ is a general position vector; in rectangular coordinates, these can be written:

$$\overline{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \qquad \overline{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\text{where} \qquad {k_x}^2 + {k_y}^2 + {k_z}^2 = k^2$$

The field strengths $\overline{E}_0^+$ and $\overline{E}_0^-$ are constant vectors, with the $+$ and $-$ superscripts indicating propagation in the forward ($+\overline{k}$) direction or reverse ($-\overline{k}$) direction, respectively.

The phasor solutions above represent uniform plane waves, since the constant phase surface, $\overline{k}\cdot\overline{r} =$ constant, defines a plane surface, and the field strength is uniform everywhere. The plane-wave solution can be substituted into Maxwell's equations to give

$$\overline{k}\times\overline{E} = \omega\mu\overline{H} \qquad \overline{k}\cdot\overline{E} = 0$$
$$\overline{k}\times\overline{H} = -\omega\epsilon\overline{E} \qquad \overline{k}\cdot\overline{H} = 0$$

Electromagnetic waves in which both the electric and magnetic fields are perpendicular to the direction of propagation are called Transverse Electromagnetic (TEM) Waves.

For a wave travelling in the $+\hat{z}$ direction in free space, the above gives

$$\overline{E} = \overline{E}_0^+ e^{-\jmath k_0 z} \quad \text{and} \quad \overline{H} = \frac{\hat{z}\times\overline{E}_0^+}{\eta_0} e^{-\jmath k_0 z}$$

and Poynting's Theorem gives

$$\text{Stored Electric Energy:} \quad w_e = \frac{1}{2}\epsilon E_0^2$$

$$\text{Stored Magnetic Energy:} \quad w_m = \frac{1}{2}\mu {H_0}^2 = \frac{1}{2}\mu\left(\frac{E_0}{\eta}\right)^2 = \frac{1}{2}\epsilon {E_0}^2$$

$$\text{Average Power Density:} \quad \overline{P}_{ave} = \frac{1}{2}\text{Re}\left\{\overline{E}\times\overline{H}^*\right\} = \frac{{E_0}^2}{2\eta}\hat{z} = \frac{1}{2}\eta {H_0}^2\hat{z}$$

In vacuum or free-space, where $\epsilon = \epsilon_0$ and $\mu = \mu_0$, $\eta_0 = 377\,\Omega$. Variables such as η_0, k_0, and λ_0 with subscripts 0 denote free-space values.

The general properties of a uniform plane wave can therefore be summarized as follows:

- Electric and magnetic field are perpendicular to each other
- No electric or magnetic field in the direction of propagation
- The value of the magnetic field is equal to the magnitude of the electric field divided by η at every instant
- The direction of propagation is in the same direction as $\overline{E} \times \overline{H}$
- The instantaneous value of the Poynting vector is given by E^2/η, or $H^2\eta$
- The average value of the Poynting vector is given by $E^2/2\eta$, or $H^2\eta/2$
- The stored electric energy is equal to the stored magnetic energy at any instant.

Propagation in Lossy Media:

Losses are incorporated using Ohm's law, which leads to a modified Helmholtz equation

$$\nabla^2\overline{E} - \gamma^2\overline{E} = 0$$

$$\text{where} \qquad \gamma^2 = -\omega^2\mu\epsilon_c \qquad \text{and} \qquad \epsilon_c = \epsilon - \jmath\frac{\sigma}{\omega} = \epsilon' - \jmath\epsilon''$$

It is customary to define an additional parameter called the loss tangent, $\tan\delta \equiv \epsilon''/\epsilon'$. The real and imaginary parts of the complex propagation constant, γ, are written as $\gamma = \alpha + \jmath\beta$. For example, a plane wave solution for a wave propagating in the $+\hat{z}$ direction in lossy media is

$$\overline{E} = \hat{x}E_x e^{-\gamma z} = \hat{x}E_x e^{-\alpha z}e^{-\jmath\beta z}$$

So α represents the attenuation rate in Nepers per meter. An attenuation of $1\,\text{Np/m}$ means that the wave decays to e^{-1} of its original value in one meter. One Np/m is equivalent to $20\log e = 8.686\,\text{dB/m}$. In addition to the general case, there are two special cases of interest:

- Low-loss dielectric: $\epsilon'' \ll \epsilon'$

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \qquad \beta \approx \omega\sqrt{\mu\epsilon} \qquad \eta_c \approx \sqrt{\frac{\mu}{\epsilon}}\left(1 + \jmath\frac{\tan\delta}{2}\right)$$

- Good conductor (high loss material): $\epsilon'' \gg \epsilon'$

$$\alpha = \beta = \sqrt{\pi f\mu\sigma} \qquad \eta_c = (1+\jmath)\frac{\alpha}{\sigma}$$

In lossy materials, particularly conductors, it is often useful to define a skin depth, or penetration depth, which is a characteristic length for the penetration of electromagnetic fields (and currents) into a material. The skin depth is defined as

$$\delta \equiv \frac{1}{\alpha} \quad [\text{m}]$$

this δ is not to be confused with the symbol appearing in the loss tangent, $\tan\delta$.

Wave Velocities:

In dispersive media, the wave velocity described above will be a function of frequency, hence signals composed of a group of different frequencies will tend to spread out, or disperse as the signal propagates. This leads to a distinction between the phase velocity, or the velocity of a monochromatic wave, and the group velocity, or the velocity of the envelope of a wavepacket:

$$\text{Phase Velocity} \quad v_p = \frac{\omega}{k}\ \text{m/s} \qquad \text{Group Velocity} \quad v_g = \frac{\partial\omega}{\partial k}\ \text{m/s}$$

In non-dispersive media, the group velocity is equal to the phase velocity.