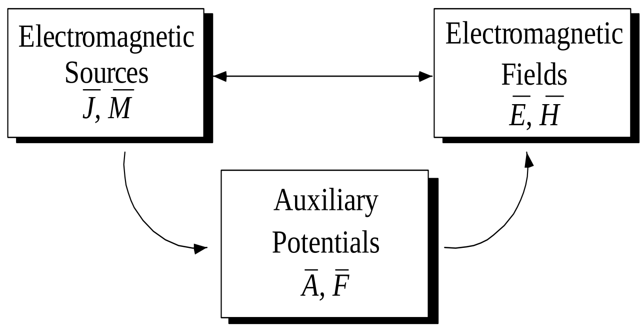
# Vector Potentials

In problems requiring the calculation of fields produced by applied currents and charge distributions, a direct analytical solution of Maxwell's equations is very difficult, and it is often advantageous to introduce a new pair of functions $\overline{A}$ and $V$, called the vector and scalar potentials, respectively. Maxwell's equations are then rewritten in terms of these potentials, which can sometimes enable us to solve problems with much less effort.




Maxwell's equations in a region containing currents and charges are:

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \nabla \cdot \overline{D} = \rho \tag{1}$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \qquad \nabla \cdot \overline{B} = 0$$

Since the divergence of any curl is identically zero, we must be able to represent the magnetic flux density $\overline{B}$ as the curl of some new function $\overline{A}$

$$\overline{B} = \nabla \times \overline{A} \tag{2}$$

Substituting this into Faraday's law gives

$$\nabla \times \overline{E} = -\frac{\partial}{\partial t}(\nabla \times \overline{A})$$

$$\text{or} \quad \nabla \times \left(\overline{E} + \frac{\partial \overline{A}}{\partial t}\right) = 0 \tag{3}$$

Since the curl of any gradient is identically zero, that means we must be able to write the contents of the expression in parentheses in equation (3) as the gradient of some scalar function $V$:

$$\overline{E} + \frac{\partial \overline{A}}{\partial t} = -\nabla V \tag{4}$$

If there are no time variations, (4) reduces to the definition of the electrostatic potential. Thus the new scalar potential which we have defined is a generalization of the electrostatic potential to time-varying fields, and this explains why the negative sign in (4) was chosen. Substituting (3) and (4) into the other two Maxwell equations gives

$$\begin{aligned} \nabla^2 \overline{A} - \mu\epsilon \frac{\partial^2 \overline{A}}{\partial t^2} &= -\mu \overline{J} + \nabla\left(\nabla \cdot \overline{A} + \mu\epsilon \frac{\partial V}{\partial t}\right) \\ \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \overline{A}) &= -\frac{\rho}{\epsilon} \end{aligned} \tag{5}$$

Helmholtz's theorem states that a vector function is uniquely specified if both its divergence and curl are defined. So far we have only defined the curl of $\overline{A}$ through (2). We are free to choose $\nabla \cdot \overline{A}$ to be anything we like, and the particular choice of $\nabla \cdot \overline{A}$ is called the gauge. One popular choice is the Lorentz Gauge:

$$\text{(Lorentz gauge)} \qquad \nabla \cdot \overline{A} \equiv -\mu\epsilon \frac{\partial V}{\partial t}$$

which decouples the two equations in (5) and gives

$$\nabla^2 \overline{A} - \mu\epsilon \frac{\partial^2 \overline{A}}{\partial t^2} = -\mu \overline{J} \qquad \nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{6}$$

These are in the form of inhomogeneous wave equations. For time-harmonic fields, equations (6) become the inhomogeneous Helmholtz equations

$$\nabla^2 \overline{A} - k^2 \overline{A} = -\mu \overline{J} \qquad \nabla^2 V - k^2 V = -\frac{\rho}{\epsilon} \tag{7}$$

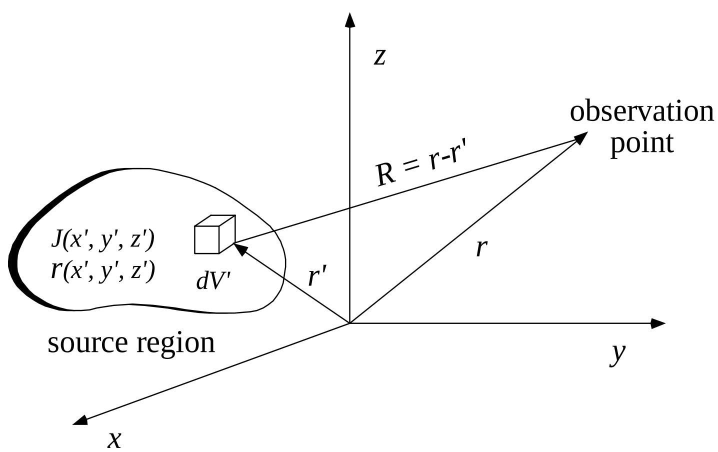
The formal solution of the wave equations in (6) are

$$\begin{aligned} \overline{A}(x,y,z,t) &= \frac{\mu}{4\pi} \int_{V'} \frac{\overline{J}(x',y',z',t-R/v)}{R}\, dV' \\ V(x,y,z,t) &= \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(x',y',z',t-R/v)}{R}\, dV' \end{aligned} \tag{8}$$

where

$$v = \frac{1}{\sqrt{\mu\epsilon}} \qquad R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

and the integration is over the volume $V'$ which contains the sources, as shown below.




Equations (8) are sometimes called the retarded potentials, because they indicate that the fields at time $t$ depend on the sources at some previous time $t - R/v$; that is, there is a finite time for the effect to propagate away from the source region $V'$. For the time-harmonic case, equations(8) become

$$\begin{aligned} \overline{A}(x,y,z) &= \frac{\mu}{4\pi} \int_{V'} \frac{\overline{J}(x',y',z')e^{-\jmath kR}}{R}\, dV' \\ V(x,y,z) &= \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(x',y',z')e^{-\jmath kR}}{R}\, dV' \end{aligned} \tag{9}$$

If the current distribution is known, either (8) or (9) allow us to calculate the vector potential $\overline{A}$, from which we can determine the magnetic field using (2). Outside of the source region, we can relate the electric and magnetic fields throuhg Maxwell's equations, so that our final solution is

$$\overline{H} = \frac{1}{\mu} \nabla \times \overline{A} \qquad \overline{E} = \frac{1}{\jmath\omega\epsilon} \nabla \times \overline{H}$$