General Waveguide Theory

Basic Equations

Consider wave propagation along the z-axis, with fields varying in time and distance according to $e^{(\jmath\omega t-\gamma z)}$. The propagation constant γ gives us much information about the character of the waves. We will assume that the fields propagating in a waveguide along the z-axis have no other variation with z, that is, the transverse fields do not change shape (other than in magnitude and phase) as the wave propagates.

Maxwell's curl equations in a source-free region ($\rho = 0$ and $\overline{J} = 0$) can be combined to give the wave equations, or in terms of phasors, the Helmholtz equations:

$$\nabla^2\overline{E} + k^2\overline{E} = 0 \qquad \nabla^2\overline{H} + k^2\overline{H} = 0$$

where $k = \omega\sqrt{\mu\epsilon}$. In rectangular or cylindrical coordinates, the vector Laplacian can be broken into two parts

$$\nabla^2\overline{E} = \nabla_t^2\overline{E} + \frac{\partial^2\overline{E}}{\partial z^2}$$

so that with the assumed $e^{-\gamma z}$ dependence we get the wave equations

$$\nabla_t^2\overline{E} + (\gamma^2 + k^2)\overline{E} = 0 \qquad \nabla_t^2\overline{H} + (\gamma^2 + k^2)\overline{H} = 0$$

Substituting the $e^{(\jmath\omega t-\gamma z)}$ into Maxwell's curl equations separately gives (for rectangular coordinates)

$$\begin{aligned} \nabla\times\overline{E} &= -\jmath\omega\mu\overline{H} & \nabla\times\overline{H} &= \jmath\omega\epsilon\overline{E} \\ \frac{\partial E_z}{\partial y} + \gamma E_y &= -\jmath\omega\mu H_x & \frac{\partial H_z}{\partial y} + \gamma H_y &= \jmath\omega\epsilon E_x \\ -\gamma E_x - \frac{\partial E_z}{\partial x} &= -\jmath\omega\mu H_y & -\gamma H_x - \frac{\partial H_z}{\partial x} &= \jmath\omega\epsilon E_y \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\jmath\omega\mu H_z & \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= \jmath\omega\epsilon E_z \end{aligned}$$

These can be rearranged to express all of the transverse field components in terms of E_z and H_z, giving

$$\begin{aligned} E_x &= -\frac{1}{\gamma^2+k^2}\left(\gamma\frac{\partial E_z}{\partial x} + \jmath\omega\mu\frac{\partial H_z}{\partial y}\right) & H_x &= \frac{1}{\gamma^2+k^2}\left(\jmath\omega\epsilon\frac{\partial E_z}{\partial y} - \gamma\frac{\partial H_z}{\partial x}\right) \\ E_y &= \frac{1}{\gamma^2+k^2}\left(-\gamma\frac{\partial E_z}{\partial y} + \jmath\omega\mu\frac{\partial H_z}{\partial x}\right) & H_y &= -\frac{1}{\gamma^2+k^2}\left(\jmath\omega\epsilon\frac{\partial E_z}{\partial x} + \gamma\frac{\partial H_z}{\partial y}\right) \end{aligned}$$

For propagating waves, $\gamma = \jmath\beta$, where β is a real number provided there is no loss. Rewriting the above for propagating waves, and using the substitution $k_c^2 \equiv \gamma^2 + k^2$, gives

$$\begin{aligned} E_x &= -\frac{\jmath}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right) & H_x &= \frac{\jmath}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right) \\ E_y &= \frac{\jmath}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right) & H_y &= -\frac{\jmath}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right) \end{aligned}$$

The analytic procedure for finding waveguide fields and propagation constants is to solve the wave equations for the z-components of the fields, subject to the boundary conditions for the waveguide, and then find the transverse field components from the above.

Mode Classification

In uniform waveguides it is common to classify the various wave solutions found from the previous analysis into the following types:

- **TEM waves**: waves with no electric or magnetic field in the direction of propagation ($H_z = E_z = 0$). Plane waves and transmission-line waves are common examples.
- **TM waves**: waves with an electric field but no magnetic field in the direction of propagation ($H_z = 0, E_z \neq 0$). These are sometimes referred to as E waves.
- **TE waves**: waves with a magnetic field but no electric field in the direction of propagation ($H_z \neq 0, E_z = 0$). These are sometimes referred to as H waves.
- **Hybrid waves**: Sometimes the boundary conditions require all field components. These waves can be considered as a coupling of TE and TM modes by the boundary.

Note that these are not the only way to categorize the different wave solutions, but have been standardized by long usage.

Mode Impedances, Propagation Constants, and Cutoff Frequencies

The various types of wave solutions have many common features, regardless of the shape of the waveguiding structure. Examining the propagation constant $\gamma = \sqrt{k_c^2 - k^2}$ we see that for some frequencies it is imaginary ($\gamma = \jmath\beta$), corresponding to propagating waves, and for others it is real ($\gamma = \alpha$), corresponding to exponentially decaying, or evanescent fields. The dividing line is at a frequency known as the cutoff frequency, given by

$$f_c = \frac{k_c}{2\pi\sqrt{\mu\epsilon}} \qquad \text{cutoff frequency}$$

we can write the propagation constant in terms of f_c as follows

$$\gamma = \begin{cases} \jmath\beta = \jmath k\sqrt{1-(f_c/f)^2} & f > f_c \\ \alpha = k_c\sqrt{1-(f/f_c)^2} & f < f_c \end{cases}$$

Thus above the cutoff frequency, waves can propagate. Note from the field expression derived previously that TEM waves can only have non-zero fields if $\gamma^2 + k^2 = 0$, or $\gamma = \jmath k = \jmath\omega\sqrt{\mu\epsilon}$. In this case, $k_c = 0$, hence TEM waves have no cutoff frequency.

From our expressions for the transverse field components we can also define wave impedances for the various modes. The wave impedance concept is important because it allows us to relate the $\overline{E}$ and $\overline{H}$ fields through the simple relationship $\overline{H} = \hat{z} \times \overline{E}/Z$, thus unifying the presentation of electromagnetic waves. And from a practical standpoint the wave impedance allows us to use transmission-line theory to describe non-TEM waveguide circuits.

Mode	Propagation Constant, β	Guide Wavelength, λ_g	Wave Impedance, Z
TEM	$k = \omega\sqrt{\mu\epsilon}$	$\lambda = 1/f\sqrt{\mu\epsilon}$	$\eta = \sqrt{\mu/\epsilon}$
TM	$k\sqrt{1-(f_c/f)^2}$	$\dfrac{\lambda}{\sqrt{1-(f_c/f)^2}}$	$\dfrac{\gamma}{\jmath\omega\epsilon} = \eta\sqrt{1-(f_c/f)^2}$
TE	$k\sqrt{1-(f_c/f)^2}$	$\dfrac{\lambda}{\sqrt{1-(f_c/f)^2}}$	$\dfrac{\jmath\omega\mu}{\gamma} = \dfrac{\eta}{\sqrt{1-(f_c/f)^2}}$

Below cutoff the wave impedance is imaginary, indicating that there is no net transfer of power. Since the propagation constants and wave impedances for non-TEM modes are nonlinear functions of frequency, such modes are dispersive.