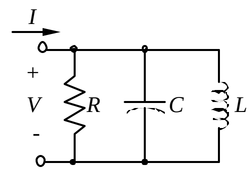


Figure 1.3 RLC circuit for interpreting Poynting's theorem.

We know from circuit theory that the real power delivered to the circuit is $P_{loss} = \frac{1}{2}RII^*$, and the energy stored in the inductor and capacitor is given by $U_m = \frac{1}{4}LII^*$ and $U_e = \frac{1}{4}CVV^*$, respectively, which enables us to write (1.35) as

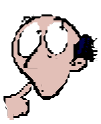
$$P_{in} = P_{loss} + 2\jmath\omega\left(U_m - U_e\right) \tag{1.36}$$

which is exactly the same form as (1.34). Furthermore, this suggests the following expressions for stored magnetic energy and stored electric energy in terms of the fields:

$$U_m = \frac{1}{4}\text{Re}\iiint \overline{H}^* \cdot \overline{B}\,dV \qquad U_e = \frac{1}{4}\text{Re}\iiint \overline{E}\cdot\overline{D}^* dV \tag{1.37}$$

It should be noted that these expressions for energy density are not valid for dispersive media, but can be suitably modified (see p.94 of [2]).

1.6 SOURCES AND GENERATORS



In formulating electromagnetic problems, we may postulate a set of charges or currents as known ***sources*** of fields, and subsequently attempt to formulate more direct solutions for the field quantities in terms of these sources. However, we must remember that the fields so produced are also capable of inducing surface charges and currents in neighboring matter. These induced currents and charges will then give rise to another set of fields which are superimposed on the first, and so on. This phenomenon is called ***scattering***, and the secondary fields produced by the induced currents are the ***scattered fields***. Although the induced charges and currents also act as sources of the scattered fields, they are clearly different than the original set of charges and currents, which were assumed to exist independent of the presence of any fields. Using the superposition principle, we therefore express the ***total*** currents, $\overline{J}$ and $\overline{M}$ in Maxwell's equations, as

$$\overline{J} = \overline{J}_i + \overline{J}_f \qquad \overline{M} = \overline{M}_i + \overline{M}_f \tag{1.38}$$

where $(\overline{J}_i, \overline{M}_i)$ are the ***impressed currents***, and $(\overline{J}_f, \overline{M}_f)$ are the currents ***induced*** by the fields. Impressed currents are assumed to be fixed in some way that is not affected by the fields; these are analogous to the ideal current generators used in circuit theory. Induced currents are those that flow only in response to the fields, and arise physically from motion of either free or bound electrons. The motion of free electrons is described classically through Ohm's law and is called a ***conduction*** current, whereas the oscillatory motion of bound charges is called a ***polarization current***.

The linearity of Maxwell's equations (assuming ***linear*** media) means that the fields can be similarly decomposed into a component due only to the impressed currents (the "applied" or "incident" field), and a component produced by the induced currents (the "scattered" field),

$$\overline{E} = \overline{E}_{\text{inc}} + \overline{E}_{\text{scatt}} \qquad \overline{H} = \overline{H}_{\text{inc}} + \overline{H}_{\text{scatt}} \tag{1.39}$$

As a example of these ideas, consider the reflection of an incident field from a perfectly conducting plane, as illustrated in figure 1.4. Mathematically, Maxwell's equations describe a self-consistent relationship between the ***total*** currents and ***total*** fields for the problem, but physically it is more appealing to consider the situation as a chain of events: the incident field is produced by an impressed source distribution $\overline{J}_i$; this incident field ***induces*** a conduction current on the surface of the conductor; the induced current then acts as a source, radiating fields which must exactly cancel the incident fields on and within the conductor, as required by the boundary conditions. In later chapters we will develop quite general methods for attacking such scattering problems based on this causal viewpoint, which can be applied to almost any problem, at least in principle.

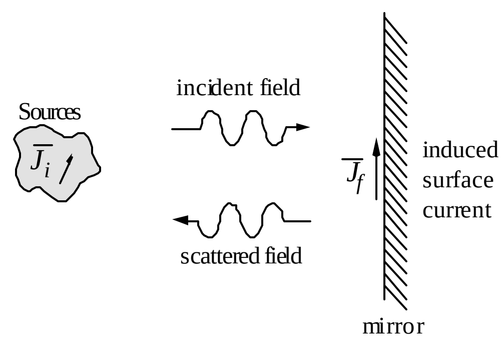


Figure 1.4 Reflection from a mirror as a simple example of scattering processes, described by impressed and induced currents.

The distinction between impressed and induced currents is therefore a natural breakdown in terms of "cause and effect", but it can sometimes lead to confusion in analysis. The trouble starts when making statements such as "currents are induced, which in turn radiate…". In order to calculate the scattered fields we may temporarily view the induced currents as fixed generators, that is, like impressed currents. In this way, the same mathematical relationship between incident fields and impressed currents can be used to relate the scattered fields to the induced currents. But it must be remembered that the currents in question are, in fact, induced currents when making such calculations. They produce only part of the field and hence must be related back to the impressed currents in such a way that all boundary conditions are satisfied. This discussion may seem rather pedantic, but a clear understanding of the differences is especially helpful in our application of field equivalence principles.

Impressed currents can be used to represent "circuit" generators as shown in fig. 1.5. A ***current source*** is modeled as a short filament of impressed current $\overline{J}_i$ in series with a perfectly conducting wire, as shown in fig. 1.5a. Assuming the dimensions of the circuit are small enough so that Kirchoff's circuit laws apply, then the impressed current will induce a current in the external circuit of the same magnitude, irrespective of the load impedance. If we compute the complex power flow out of a volume surrounding the

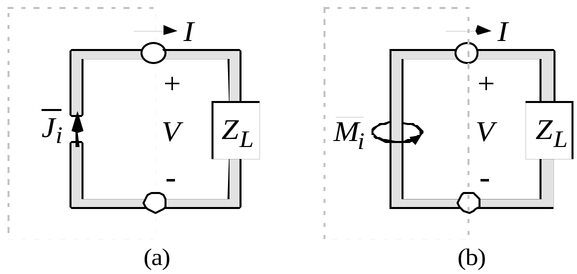


Figure 1.5 Electromagnetic representation of independent circuit sources. (a) Current generator (impressed electric current filament); (b) Voltage generator (impressed magnetic current loop).

generator (the dashed box in fig. 1.5), we find

$$-\frac{1}{2}\iiint \overline{E}\cdot\overline{J}_i^*\,dV = -\frac{1}{2}I^*\int_{gap}\overline{E}\cdot d\overline{\ell} = \frac{1}{2}I^*V \tag{1.40}$$

which is in accordance with our expectations of a current generator. Note that the internal impedance of the source is infinite, since removal of the impressed current leaves an open circuit in the gap. Similarly, a ***voltage source*** in circuit theory can be represented as in fig. 1.5b, using a filamentary loop of magnetic current around a perfectly conducting wire. From Maxwell's equations, and neglecting the magnetic flux linked by the circuit,

$$\oint_C \overline{E}\cdot d\overline{\ell} = \iint \overline{M}\cdot d\overline{S}$$

If the path C is coincident with the wire leads and closes across the terminals, then we find the magnitude of the magnetic current filament is just $-V$, and therefore the complex power flowing out of the generator (through the dashed box in fig. 1.5b) is

$$-\frac{1}{2}\iiint \overline{H}^*\cdot\overline{M}\,dV = \frac{1}{2}V\oint_{\text{loop}}\overline{H}^*\cdot d\overline{\ell} = \frac{1}{2}VI^* \tag{1.41}$$

The internal impedance in this case is zero since removal of the current loop leaves a short circuit.

1.7 RECIPROCITY THEOREMS; RUMSEY'S REACTION

Suppose there are two separate source distributions, $(\overline{J}_1, \overline{M}_1)$ and $(\overline{J}_2, \overline{M}_2)$, in a certain localized region defined by volume V, as shown in figure 1.6. Physically this situation is representative of a general two-port electrical network, such as an antenna link. Characterization of this electrical network involves examining the interaction of fields and sources between the ports. We assume the volume is filled with a simple isotropic media described by (1.13) and (1.11). These sources produce the fields $(\overline{E}_1, \overline{H}_1)$ and $(\overline{E}_2, \overline{H}_2)$, respectively, in accordance with Maxwell's equations

$$\begin{array}{ll} \nabla \times \overline{E}_1 = -\jmath\omega\mu\overline{H}_1 - \overline{M}_1 \\ \nabla \times \overline{H}_1 = \quad \jmath\omega\epsilon\overline{E}_1 + \overline{J}_1 \end{array} \quad \text{and} \quad \begin{array}{ll} \nabla \times \overline{E}_2 = -\jmath\omega\mu\overline{H}_2 - \overline{M}_2 \\ \nabla \times \overline{H}_2 = \quad \jmath\omega\epsilon\overline{E}_2 + \overline{J}_2 \end{array} \tag{1.42}$$

Using the vector identity (A.47)

$$\nabla \cdot (\overline{A} \times \overline{B}) = (\nabla \times \overline{A}) \cdot \overline{B} - (\nabla \times \overline{B}) \cdot \overline{A}$$

we find that

$$\nabla \cdot \left(\overline{E}_1 \times \overline{H}_2 - \overline{E}_2 \times \overline{H}_1\right) = \overline{J}_1 \cdot \overline{E}_2 - \overline{J}_2 \cdot \overline{E}_1 + \overline{M}_2 \cdot \overline{H}_1 - \overline{M}_1 \cdot \overline{H}_2 \tag{1.43}$$

Integrating (1.43) over the volume V and using the divergence theorem (A.54) gives

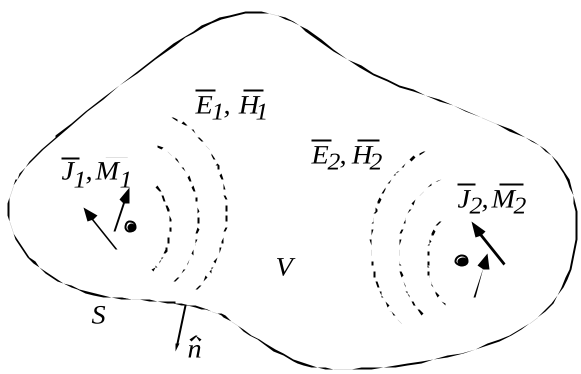


Figure 1.6 Two source distributions and corresponding fields within a volume V.

$$\oiint_S \left[\overline{E}_1 \times \overline{H}_2 - \overline{E}_2 \times \overline{H}_1\right] \cdot d\overline{S} = \iiint_V \left[\overline{J}_1 \cdot \overline{E}_2 - \overline{J}_2 \cdot \overline{E}_1 + \overline{M}_2 \cdot \overline{H}_1 - \overline{M}_1 \cdot \overline{H}_2\right] dV \tag{1.44}$$

Note that the only currents on the right hand side which contribute to the integral are the ***impressed*** sources; induced current terms cancel by virtue of (1.11). This result is usually applied (and more easily interpreted) for certain special cases where either the surface integral or volume integral vanishes. For example, if the surface is chosen to exclude any impressed sources so that $\overline{J}_1 = \overline{J}_2 = \overline{M}_1 = \overline{M}_2 = 0$, then (1.44) reduces to

$$\oiint_S \left[\overline{E}_1 \times \overline{H}_2 - \overline{E}_2 \times \overline{H}_1\right] \cdot d\overline{S} = 0 \tag{1.45}$$

Note conditions of validity!

which is called the ***Lorentz reciprocity theorem***. In this case the fields are due to sources external to S. We will later use this result to establish the reciprocal properties of an antenna link.

Alternatively, if the surface S coincides with a PEC or PMC boundary, then the surface integral vanishes since, using (A.38),

$$\begin{aligned}\left(\overline{E}_1 \times \overline{H}_2\right) \cdot \hat{n} - \left(\overline{E}_2 \times \overline{H}_1\right) \cdot \hat{n} &= \left(\hat{n} \times \overline{E}_1\right) \cdot \overline{H}_2 - \left(\hat{n} \times \overline{E}_2\right) \cdot \overline{H}_1 \\ &= -\left(\hat{n} \times \overline{H}_2\right) \cdot \overline{E}_1 + \left(\hat{n} \times \overline{H}_1\right) \cdot \overline{E}_2\end{aligned}$$

and either $\hat{n} \times \overline{E} = 0$ for a PEC boundary, or $\hat{n} \times \overline{H} = 0$ on a PMC boundary. Then (1.44) reduces to

Note conditions of validity!

$$\iiint \left(\overline{E}_1 \cdot \overline{J}_2 - \overline{H}_1 \cdot \overline{M}_2\right) dV = \iiint \left(\overline{E}_2 \cdot \overline{J}_1 - \overline{H}_2 \cdot \overline{M}_1\right) dV \qquad (1.46)$$

This is a more familiar statement of reciprocity for those knowledgeable in circuit theory, and is often simply referred to as ***the reciprocity theorem***. This last result can also be obtained if S is taken as a sphere at infinity. Then the Sommerfeld radiation condition (see Chapter 2) insures that the fields produced by localized currents in V will be spherical outward waves at infinity, so that

$$\overline{H} = \frac{\hat{n} \times \overline{E}}{\eta} \qquad \Rightarrow \qquad (\hat{n} \times \overline{E}_1) \cdot \overline{H}_2 - (\hat{n} \times \overline{E}_2) \cdot \overline{H}_1 = 0.$$

Interestingly, many of the "physical observables" important in applied electromagnetics—that is, quantities that can be measured directly—can be expressed in terms of integrals like those in (1.46). Rumsey [6] has argued for the physical significance of these integrals, which he called ***reaction integrals***. The left hand side of (1.46) is then thought of as the ***reaction*** of source #2 on the fields from source #1. This refers to the fact that, in order to keep flowing, the sources must "react" to the fields in their vicinity by supplying/absorbing energy. Reaction integrals are commonly abbreviated as

$$< i, j > = \iiint \left(\overline{E}_i \cdot \overline{J}_j - \overline{H}_i \cdot \overline{M}_j\right) dV \qquad (1.47)$$

The reciprocity theorem (1.46) can then be represented concisely as

$$< i, j > = < j, i > \qquad (1.48)$$

An important problem for later work that can be described in terms of reaction integrals is the determination of equivalent circuit parameters representing a multiport electromagnetic network, as shown in figure 1.7. Using fig. 1.7a, the impedance matrix is defined by

$$\bar{V} = \overline{\overline{Z}} \cdot \bar{I} \qquad \text{or} \qquad V_i = \sum_{j=1}^{N} Z_{ij} I_j \qquad (1.49)$$

Each term in the summation, $Z_{ij} I_j$, gives the contribution to the terminal voltage—the induced EMF—at port i due to currents impressed at port j, with all other ports open-circuited. Assuming that the independent current sources of fig. 1.7a are implemented in the sense of fig. 1.5a, we can compute the reaction $< j, i >$ as

$$< j, i > = \iiint_{\text{port } i} \overline{E}_j \cdot \overline{J}_i \, dV = I_i \int_{\text{port } i} \overline{E}_j \cdot d\bar{\ell} = -I_i \left(Z_{ij} I_j\right) \qquad (1.50)$$

where the last equality follows since the path integral or $\overline{E}_j$ over port i is just the voltage induced at port i due to the current source at port j, or $Z_{ij}I_j$. Therefore,

$$Z_{ij} = -\frac{<j,i>}{I_i I_j} = -\frac{1}{I_i I_j}\iiint \overline{E}_j \cdot \overline{J}_i \, dV \tag{1.51}$$

Using the reciprocity theorem (1.46) we find that

$$Z_{ij} = Z_{ji}$$

which is the familiar result from circuit theory. If we had alternatively chosen to express

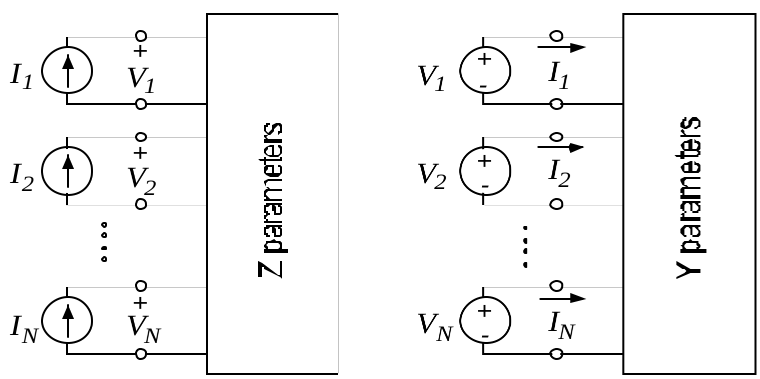


Figure 1.7 Multiport circuit representation of an electromagnetic system. (a) Configuration for characterization in terms of impedance parameters; (b) Configuration for characterization in terms of admittance parameters.

the system in terms of admittance parameters as shown in figure 1.7b, a similar analysis gives

$$Y_{ij} = \frac{<j,i>}{V_i V_j} = \frac{1}{V_i V_j}\iiint \overline{H}_j \cdot \overline{M}_i \, dV \tag{1.52}$$

with a similar consequence of reciprocity,

$$Y_{ij} = Y_{ji}$$

Closer examination of the derivation of (1.44) shows that it is critically dependent on the assumption of a simple isotropic media in the volume V. That is, we have only proved reciprocal properties for electrical systems comprised on isotropic media. Using more general constitutive relations for anisotropic media (1.14) and (1.15), we find (Problem 1.3) that the result (1.44) is only obtained when the material properties in the volume are described by symmetric dyads

$$\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}^T \qquad \overline{\overline{\mu}} = \overline{\overline{\mu}}^T \qquad \overline{\overline{\sigma}} = \overline{\overline{\sigma}}^T \tag{1.53}$$

Such materials are therefore called reciprocal materials. An important example of a material that is not reciprocal is a magnetically-biased plasma, such as the Earth's ionosphere. Antenna links involving propagation through the Earth's ionosphere are therefore not reciprocal. Alternatively, antennas themselves may be constructed from non-reciprocal media, such as magnetically-biased ferrites. The resulting non-reciprocal antenna may serve a useful function; for example, simultaneously transmitting and receiving different polarizations. Propagation in non-reciprocal media and analysis of non-reciprocal antennas is, however, a relatively specialized topic that will not be dealt with in this work.

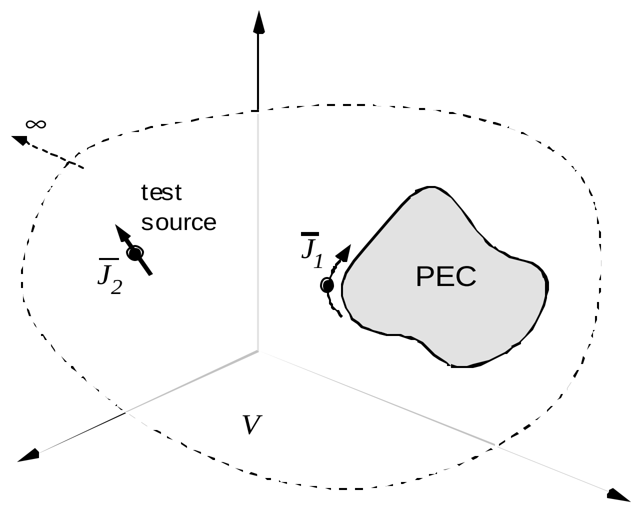


Figure 1.8 Impressed currents above conductors do not radiate, as can be shown by aplying the reciprocity theorem to this example.

The reciprocity theorem will prove a useful tool in other contexts. For example, consider the situation in fig. 1.8, where there are two sets of impressed currents, denoted as $\overline{J}_1$ and J_2, within a volume V. Current $\overline{J}_1$ is impressed directly adjacent to a PEC object. Current $\overline{J}_2$ is a test source that can be oriented in any arbitrary direction. According to (1.46), these currents and the fields they produce are related by

$$\iiint_v \overline{E}_1 \cdot \overline{J}_2 \, dV = \iiint_V \overline{E}_2 \cdot \overline{J}_1 \, dV \tag{1.54}$$

Now, the field $\overline{E}_2$ is the total field produced by the test source $\overline{J}_2$, which must vanish everywhere along the surface of the PEC object. Therefore the integral on the right in (1.54) is zero and we have

$$\iiint_V \overline{E}_1 \cdot \overline{J}_2 \, dV = 0$$

Since $\overline{J}_2$ can be anything we choose, this must mean that $\overline{E}_1 = 0$ ***everywhere*** inside of V. This proves that impressed electric currents on PEC surfaces do not radiate. Physically this is because the induced currents on the object radiate fields which exactly cancel the fields of the impressed current.

The method just employed is quite powerful. Think of what we have just done—we've solved for the fields produced by an arbitrary current distribution $\overline{J}_1$ radiating in the presence of a conductor, an otherwise difficult boundary-value problem. All that was necessary was a knowledge of the fields produced by our "testing" source, which can be anythign we choose.

1.8 UNIQUENESS OF SOLUTIONS

After going to the trouble of finding a solution to Maxwell's equations for a particular problem, one may wonder if it is the only possible solution. This is guaranteed, under certain conditions, by the uniqueness theorem. To prove the theorem, we assume the existence of two possible solutions, and derive the conditions required to insure they are identical.

Let $(\overline{E}_1, \overline{H}_1)$ and $(\overline{E}_2, \overline{H}_2)$ be two possible solutions to (1.5) for a given set of sources,

$$\begin{aligned} \nabla \times \overline{E}_1 &= -\overline{M} - \jmath\omega\mu\overline{H}_1 \qquad & \nabla \times \overline{H}_1 &= \overline{J} + \jmath\omega\epsilon\overline{E}_1 \\ \nabla \times \overline{E}_2 &= -\overline{M} - \jmath\omega\mu\overline{H}_2 \qquad & \nabla \times \overline{H}_2 &= \overline{J} + \jmath\omega\epsilon\overline{E}_2 \end{aligned} \tag{1.55}$$

Subtracting these equations and defining the difference fields $\delta\overline{E} = \overline{E}_1 - \overline{E}_2$ and $\delta\overline{H} = \overline{H}_1 - \overline{H}_2$ gives

$$\nabla \times \delta\overline{E} = -\jmath\omega\mu\delta\overline{H} \qquad \nabla \times \delta\overline{H} = \jmath\omega\epsilon\delta\overline{E} \tag{1.56}$$

which are just the source-free Maxwell equations. Therefore, the difference fields must satisfy Poynting's theorem (1.34),

$$\oint\!\!\!\oint_S (\delta\overline{E} \times \delta\overline{H}^*) \cdot d\overline{S} = \jmath\omega \iiint \left[\mu|\delta\overline{H}|^2 - \epsilon^*|\delta\overline{E}|^2\right] \, dV \tag{1.57}$$

If the solution were indeed unique, then this would imply that $\delta\overline{E} = \delta\overline{H} = 0$ everywhere within the volume of interest, so that both sides of (1.57) vanishes. Suppose we now reverse the problem: if we can somehow prove that the surface integral in (1.57) vanishes, under what conditions does this imply that the solution is unique? Expanding the volume integral in terms of its real and imaginary components, a vanishing surface integral would require that

$$\iiint_V \left[\mu'|\delta\overline{H}|^2 - \epsilon'|\delta\overline{E}|^2\right] \, dV = 0 \tag{1.58a}$$

$$\iiint_V \left[\mu''|\delta\overline{H}|^2 + \epsilon''|\delta\overline{E}|^2\right] \, dV = 0 \tag{1.58b}$$

In lossy media, μ'' and ϵ'' are always positive. As long as there is some finite (though perhaps infinitesimal) loss in the system, the second of (1.58) can only be satisfied if $\delta\overline{E} = \delta\overline{H} = 0$ everywhere in the volume. Since there is always *some* loss in practice, uniqueness is therefore guaranteed provided we can make the surface integral vanish.

Using (A.38) and $d\overline{S} = \hat{n}\, dS$, the integrand of the surface integral in (**??**) can be written as

$$(\delta\overline{E} \times \delta\overline{H}^*) \cdot d\overline{S} = (\hat{n} \times \delta\overline{E}) \cdot \delta\overline{H}^* \, dS = -(\hat{n} \times \delta\overline{H}^*) \cdot \delta\overline{E} \, dS \tag{1.59}$$

If the tangential electric fields are specified on the bounding surface—for example, if the problem statement fixes the value of $\hat{n} \times \overline{E}$ on S—then this boundary condition must be incorporated into every possible solution, hence $\hat{n} \times \delta\overline{E} = 0$ over S, and the surface integral vanishes. Similarly, if the tangential magnetic fields, $\hat{n} \times \overline{H}$), are specified on the surface, then $\hat{n} \times \delta\overline{H} = 0$ over S, and the surface integral vanishes. Therefore, ***the fields produced by sources within a lossy region are unique as long as the tangential components satisfy prescribed conditions at the bounding surface.*** To obtain uniqueness in an ideal lossless region, we consider the fields to be the limit of a corresponding field in a lossy region as the loss goes to zero [4].

1.9 FIELD EQUIVALENCE PRINCIPLES

It many problems, a knowledge of the fields is required not everywhere in space, but rather in a certain well-defined region that is separate from the sources of the field (for example, the radiation fields of an antenna). In such cases it may be possible to simplify the problem by replacing the actual sources with fictitious sources that produce the same fields in the region of interest. These provide powerful tools for analysis.

1.9.1 Image Theory

The simplest and most familiar equivalence is the method of images. This technique is really just a catalog of certain electromagnetic problems that produce identical field distributions. These are usually identified by noting that conducting surfaces are surfaces of constant potential, and therefore can be placed along equipotential lines in ***any*** field distribution without altering the fields. For example, in fig. 1.9a, the fields produced by a positive and negative charge separated by a distance $2x$ produce an equipotential surface midway between the two charges. If we place a conducting object along this equipotential surface as shown in fig. 1.9b, then the fields above the surface are unchanged.

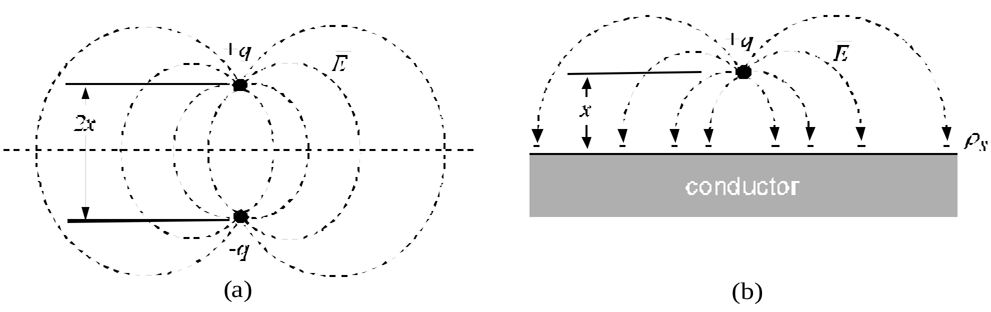


Figure 1.9 (a) Positive and negative charges separated by $2x$. (b) Positive charge $+q$ a distance x above a ground plane. These two situations are identical as far as the fields above the ground plane are concerned.

This equivalence is usually applied in reverse. Given a situation where there are charges a distance x a conducting plane, the conductor can be replaced by a set of image charges that have the opposite sign as the original charge, spaced a distance x below the original conducting surface. This eliminates the conducting matter, leaving only charges in unbounded space, a considerably easier problem to solve. Note, however, that this equivalence applies only to the fields above the original conductor.

From this simple example we can derive many other image equivalents involving electric currents, and magentic charges and currents, and PMC surfaces. These are summarized in fig. 1.10 below. Note that the images for current elements depend on the direction of the current element, and can be derived from a knowledge of the behavior of the image charges as they are moved relative to the conducting surface. For example, a horizontal current element $\overline{J}_h$ above a PEC ground plane corrsponds to a positive charge movement in the direction of the current flow. The image charge in this case would move in the same direction, but has the opposite sign so that the effective image current direction is reversed.

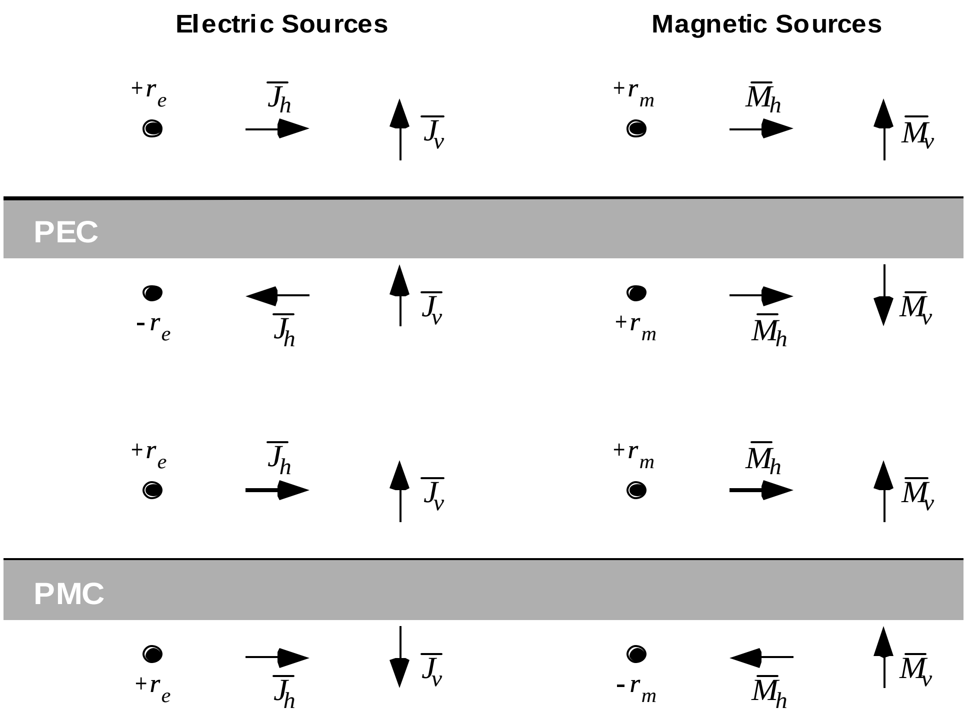


Figure 1.10 Summary of equivalent images for sources near conductors.

1.9.2 Love-Shelkunoff Equivalents

Readers familiar with circuit theory will remember that a network containing sources which drives a passive load network can be replaced by a Thevenin or Norton equivalent. This is illustrated in figure 1.11. Insofar as the calculation of voltage and current in the load network is concerned, the original and equivalent sources behave the same.

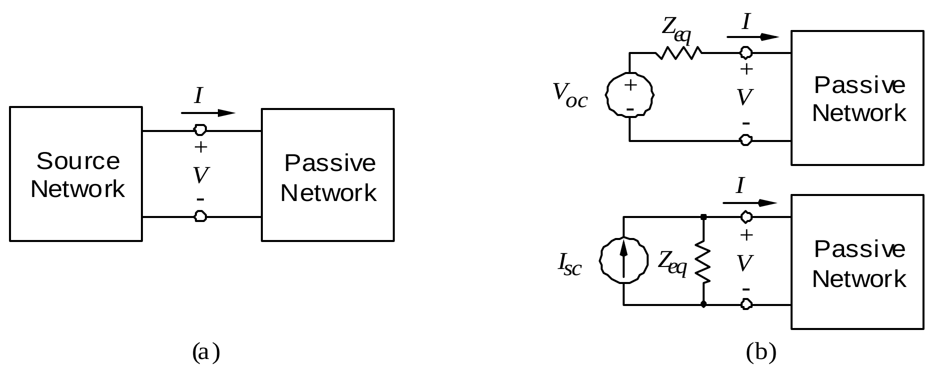


Figure 1.11 Thevenin and Norton equivalence principles from circuit theory.

In the case of the Thevenin and Norton circuits, the equivalent sources are expressed in terms of the open circuit voltage, V_{oc}, and short-circuit current, I_{sc}, which are measured at the terminals of the source network. The equivalent source impedance in each case, Z_{eq}, is the impedance of the original source network with all the sources shut off. However, the Thevenin and Norton sources are not the only possible equivalents. In order to relate the concept more directly to field theory, it is desirable to express the equivalent sources in terms of the actual terminal voltage and current in the original problem of figure 1.11. This is accomplished by the equivalent circuits of figure 1.12. These circuits are equivalent to the original problem because the relationship between V and I is fixed by the passive load network, which remains unchanged. Simple analysis shows that there is no current flowing in the source impedance of figure 1.12a, so it can be specified arbitrarily; figures 1.12b and 1.12c result from the choice $Z_{eq} = 0$ and $Z_{eq} = \infty$, respectively.

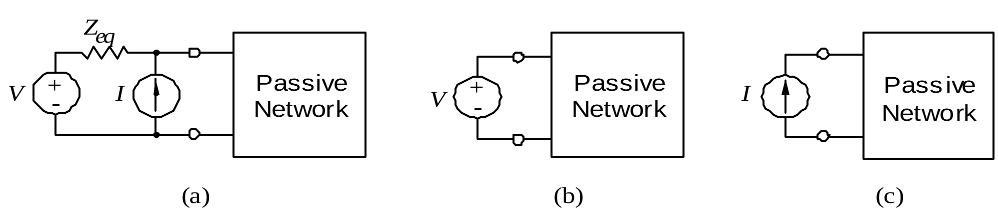


Figure 1.12 Other possible equivalents sources in terms of the actual terminal voltage V and current I from the previous figure.

The same concepts are extended to field theory by considering the situation depicted in figure 1.13a. Sources within some bounded region, possibly containing matter, produce the fields $(\overline{E}, \overline{H})$ outside of that region. To simplify the calculation of these fields, we replace the original sources by ***impressed*** surface currents J_s and M_s flowing on the boundary of the source region. From (1.22), we know that the magnitude of the surface currents required to produce the same fields outside of the boundary depends on the difference of the tangential fields across the boundary. Since the region within the boundary is of no interest, we can arbitrarily specify that the fields are zero within that region, giving the equivalent of figure 1.13b. This is known as ***Love's equivalence***

principle [4]. Note that this equivalence is only helpful when the tangential fields at the boundary of the original problem are known (or can be approximated). Comparing this situation to the circuit model of figure 1.12a, we see that the magnetic current is analogous to the terminal voltage V, while the electric current is analogous to the terminal current, I.

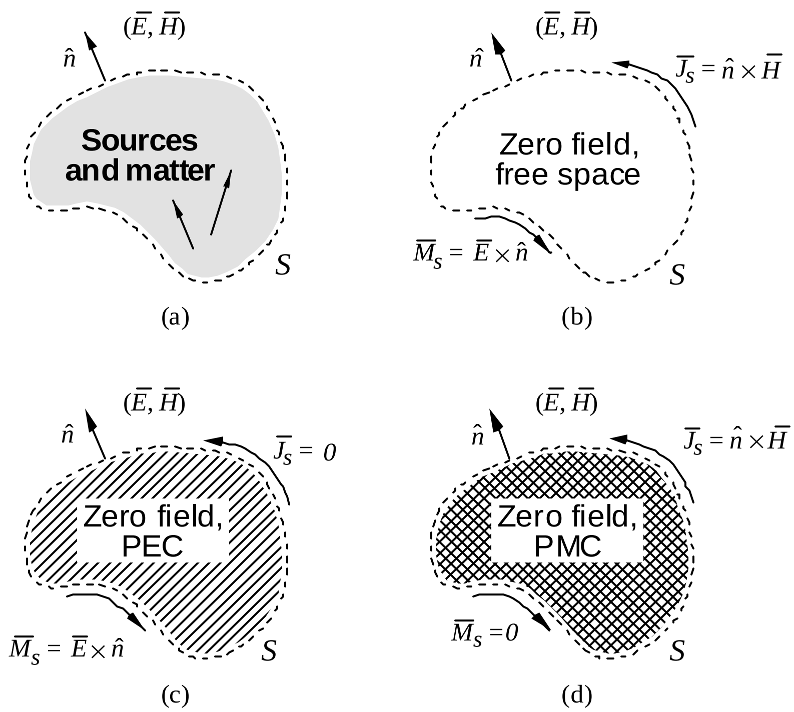


Figure 1.13 Four possible source configurations which produce the same field configuration external to the boundary S. (a) Original problem. (b) Love's equivalent, where the original source region is replaced by free-space, and surface currents are impressed on the bounding surface to produce null field within S. (c) and (d) are Shelkunoff equivalents, where the original source region is replaced by a perfect conductor. In the latter case, the impressed currents induce additional currents on the conductors.

Since the null field was specified within the original source region, the material found within the source region is irrelevant to the calculation of fields external to that region. This is analogous to the arbitrariness of the source impedance in the circuit equivalent of figure 1.12a. The most common choices are to fill the volume with free-space, as was tacitly assumed in Love's equivalent, or to surround the region by a perfect electric conductor or perfect magnetic conductor, as shown in figures 1.13c, and 1.13d. The latter two choices are due to Shelkunoff [4], and are analogous to the circuit models of figures 1.12b and 1.12c. In the first case (figure 1.13c) only magnetic currents are required, since the impressed current J_s is "short-circuited" by the PEC and does not radiate (proved earlier using the reciprocity theorem). Similarly, the magnetic current M_s

is short-circuited in figure 1.13d, and only the electric surface current is required. From the uniqueness theorem we are assured that the fields calculated in each case will be identical to the original problem, as only one of the tangential fields ($\overline{E}$ or $\overline{H}$) is required.

Although it may not be immediately obvious, the equivalence of the four physical situations depicted in figure 1.13 can greatly simplify radiation problems. A complicated boundary-value problem (ie. sources radiating in the presence of nearby objects, such as figures 1.13a, 1.13c, 1.13d) can be reduced to an equivalent set of currents radiating in a homogeneous unbounded medium (figure 1.13b). Alternatively, we will use the equivalence of figures 1.13a, 1.13c, and 1.13d in our formulation of Huygen's principle in Chapter 2, which reduces a complicated source distribution (1.13a) to a (hopefully) simpler surface current on a conductor (1.13c or 1.13d).

1.9.3 Volume Equivalence Theorem

The volume equivalence theorem is based on the following observation: for any material body characterized by a simple scalar permittivity and permeability as in (1.13), we can write Maxwell's curl equations as

$$\nabla \times \overline{H} = \overline{J}_i + \jmath\omega\epsilon\overline{E} = \overline{J}_i + \overline{J}_p + \jmath\omega\epsilon_0\overline{E} \qquad (1.60a)$$

$$\nabla \times \overline{E} = -\overline{M}_i - \jmath\omega\mu\overline{H} = -\overline{M}_i - \overline{M}_p - \jmath\omega\mu_0\overline{H} \qquad (1.60b)$$

where we have defined the ***polarization currents***

$$\overline{J}_p = \jmath\omega(\epsilon - \epsilon_0)\overline{E} \qquad \overline{M}_p = \jmath\omega(\mu - \mu_0)\overline{H} \qquad (1.61)$$

In other words, we can replace the material by a volume current distribution (J_p, M_p) flowing in free space. The magnitude of the polarization currents are dependent on the ***total*** fields, and the total fields include some contribution from the polarization currents, so this approach is typically used to set up a self-consistent integral equation for the currents.

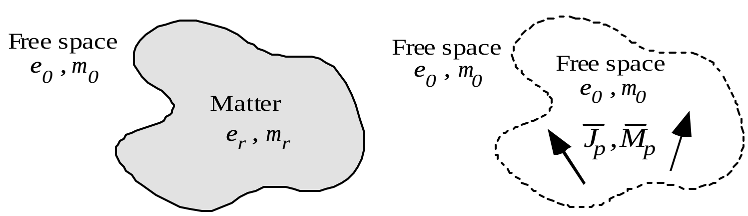


Figure 1.14 Volume equivalence theorem. Simple dielectric or magnetic matter can be replaced by volume polarization currents flowing in free-space.

1.10 FIELD BEHAVIOR NEAR A SHARP EDGE

At sharp edges of material bodies, the charge and/or current density may be highly concentrated, and in fact can become infinite in the limit of a mathematically perfect edge.

Consequently, some of the field components may also be highly peaked near an edge. It is important to understand the mathematical nature of this possible singularity, especially in numerical computation where anticipating the correct form of the fields can often greatly speed the convergence to an accurate result.

As a simple and practical example, consider the two-dimensional conducting wedge in fig. 1.15, which is infinite in the $\hat{z}$ direction. A TE-to-z excitation as shown will induce $\hat{z}$-directed currents on this object, and the associated current density function will have a mathematical singularity at the edge. From the boundary conditions, this implies that the $\hat{\rho}$ and $\hat{\phi}$ components of the scattered magnetic field will be singular at the edge. This is the only solution as guaranteed by the uniqueness theorem. Now, infinite field quantities

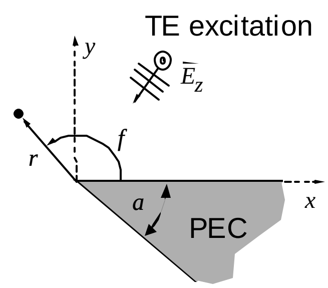


Figure 1.15 Cross section of a PEC wedge, with TE excitation.

may be acceptable as long as the ***physical observables*** derived from them remain finite. If Maxwell's equations are correct, the singular solution must therefore behave such that the ***field energy*** remain finite. This physical constraint gives us the critical information for predicting the behavior of the fields close to an edge. In the present case, the electric stored energy per unit length in a cylinder of radius a surrounding the edge is

$$\mathcal{U}_e = \frac{1}{2}\epsilon_0 \int_0^{2\pi-\alpha} \int_0^a |E_z|^2 \, \rho \, d\rho \, d\phi \tag{1.62}$$

A similar expression describes the magnetic stored energy. The fields within the fictitious cylinder can be expressed as a power series in ρ; the dominant term in the series (for small ρ) will have the form ρ^γ, where γ is the exponent to be determined. For the electric energy in (1.62) to remain finite with $E_z \propto \rho^\gamma$ requires that $\gamma > -1$. From Maxwell's equations, the dominant term in the expansion for the magnetic field can be determined as $\overline{H} \propto \nabla \times \overline{E} \propto \rho^{\gamma-1}$, and therefore for the magnetic stored energy per unit length to remain finite requires that $\gamma > 0$. Clearly then γ must be positive in order to have a finite total field energy within the cylinder. This also agrees with our expectation that $\overline{E}_z$ must vanish as $\rho \rightarrow 0$.

Near the edge we can write $\overline{E} = E_z \approx \rho^\gamma f(\phi)$. Substituting into Maxwell's equations gives

$$\frac{\partial^2 f}{\partial \phi^2} + \left(\gamma^2 + k^2\rho^2\right) f = 0. \tag{1.63}$$

Electrically close to the edge where $k\rho \ll \gamma$ we can neglect the term $k^2\rho^2$ and hence

$$f(\phi) \approx A \sin \gamma\phi + B \cos \gamma\phi \tag{1.64}$$

The unknown coefficients and allowed values of γ can be determined by the requirement that E_z vanish at $\phi = 0$ and $\phi = 2\pi - \alpha$, which gives $B = 0$ and $\gamma = n\pi/(2\pi - \alpha)$ where $n = 1, 2, \ldots$. The smallest positive γ describes the dominant term in the power series for the fields near the edge (*i.e.* for small ρ) , so we have

$$\textbf{TE case:} \quad E_z \approx A\rho^\gamma \sin\gamma\phi \qquad \text{where} \qquad \gamma = \frac{\pi}{2\pi - \alpha} \tag{1.65}$$
$$\overline{H} = -\tfrac{1}{\jmath\omega\mu}\nabla \times \overline{E} = -\frac{A}{\jmath\omega\mu}\frac{\gamma}{\rho^{1-\gamma}}\left[\hat{\rho}\cos\gamma\phi - \hat{\phi}\sin\gamma\phi\right]$$

A similar derivation can be carried out for a TM excitation (Problem 1.5), with the result

$$\textbf{TM case:} \quad H_z \approx B\rho^\gamma \cos\gamma\phi \tag{1.66}$$
$$\overline{E} = \tfrac{1}{\jmath\omega\epsilon}\nabla \times \overline{H} = \frac{B}{\jmath\omega\epsilon}\frac{\gamma}{\rho^{1-\gamma}}\left[-\hat{\rho}\sin\gamma\phi - \hat{\phi}\cos\gamma\phi\right]$$

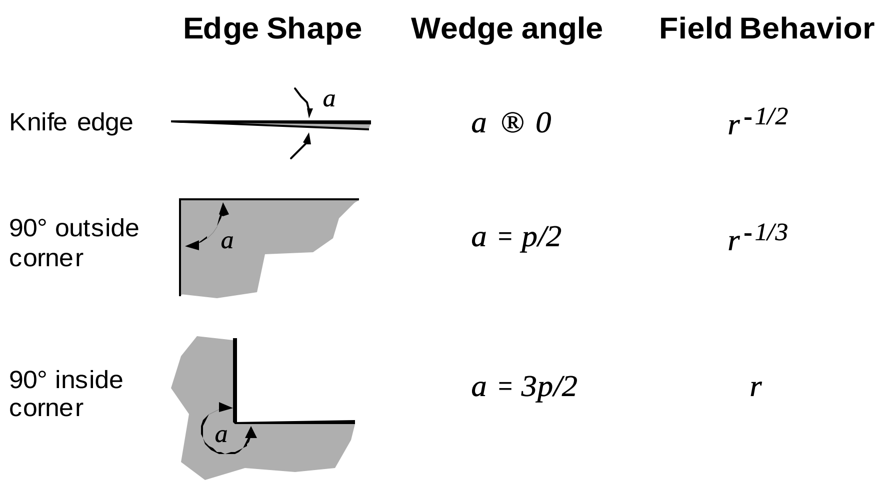


Figure 1.16 Examples of frequently encountered edges and associated singular field behavior.

So in both cases the singular fields vary as $\rho^{\gamma-1}$, where $\gamma = \pi/(2\pi - \alpha)$. Figure 1.16 illustrates three special cases often encountered in problems, and the anticipated field behavior close to the edge. Numerous other cases and treatments of singular fields can be found in the excellent monograph by Van Bladel[7].

REFERENCES

1. L. Eyges, *The Classical Electromagnetic Field*, Addison-Wesley: New York, 1972. Also reprinted by Dover, 1980.