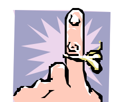
and in free space where $\eta = 120\pi$,

$$R_{rad} = 320\pi^6 \left(\frac{a}{\lambda}\right)^4$$

The radiation resistance of the loop can be extremely small, going as $(a/\lambda)^4$, compared with $(d\ell/\lambda)^2$ for the Hertzian dipole. This number can be significantly increased, however, by using a large number of turns in the loop. As long as the total coil length remains electrically small, our previous analysis applies by replacing I with (NI), where N is the number of turns. Therefore

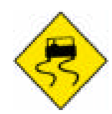
$$R_{rad} \Rightarrow 320\pi^6 N^2 (a/\lambda)^2 \tag{8.55}$$



This is valid provide that $2\pi a N \ll \lambda$.

8.4 FIELDS IN BOUNDED REGIONS

Until now we have considered currents radiating in unbounded space. More likely the currents are radiating in the presence of material objects (conductors, dielectrics) which will influence the field. In such cases the problems are usually sufficiently complicated that a numerical evaluation of the currents is required. One method for doing this is the so-called “integral equation” approach, where the fields produced by a set of unknown currents (expressed as an integral) are forced to satisfy the relevant boundary-conditions. A self-consistent solution for the currents is then sought to satisfy these boundary conditions. For this purpose, the field expressions involving the currents must be generalized from the previous sections to include the effects of the materials, *i.e.* the scattered fields.



8.4.1 Stratton-Chu Formulation

Consider the situation in fig. 8.5, where there are a number of impressed sources radiating in space, in the presence of a material object . We enclose some fraction of the sources in a volume V defined by the surface S, begin careful to exclude the material object as shown. The volume is otherwise filled with a homogeneous, isotropic medium. Green’s theorem—in one of its many forms—can then be used to relate the fields *within* the volume, to the sources in V and the fields on the surface S.

As outlined in Problem 8.14, the following identity is a version of Green’s theorem that can be derived from the vector Green’s theorem (A.65),

$$\iiint_V \left[\overline{A}\nabla^2\psi + \nabla\psi\left(\nabla\cdot\overline{A}\right) + \psi\nabla\times\nabla\times\overline{A}\right] dV$$
$$= \oiint_S \left[\psi\hat{n}\times\nabla\times\overline{A} + \left(\hat{n}\cdot\overline{A}\right)\nabla\psi + \left(\hat{n}\times\overline{A}\right)\times\nabla\psi\right] dS \tag{8.56}$$

which applies when $\overline{A}$ and ψ are continuous through the second derivative. Now, if we want this identity to give us a useful result for the fields, then clearly $\overline{A}$ should represent one of the field quantities—it doesn’t really matter which one, lets choose $\overline{A} = \overline{E}$.

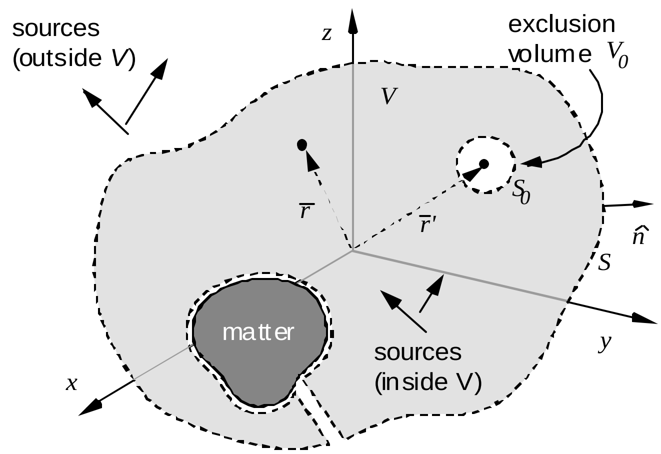


Figure 8.5 Bounded region containing some sources and excluding others. The observation point is excluded from the volume V to avoid the Green's function sinularity at $\overline{r} = \overline{r}'$.

Mathematically the function ψ can be anything, but physically the fields produced by sources are linked by a Green's function, so we choose $\psi = g$. These functions satisfy the following equations

$$\nabla \times \nabla \times \overline{E} - k^2 \overline{E} = -\jmath\omega\mu\overline{J} - \nabla \times \overline{M} \tag{8.57a}$$

$$\nabla^2 g + k^2 g = -\delta(\overline{r} - \overline{r}') \tag{8.57b}$$

There is a potential problem with the choice of $\psi = g$: the Green's function always has a discontinuity in its second derivative, at the location $\overline{r} = r'$. We can still use the theorem, but we must be sure to exclude this point from the region of integration. Physically the location $\overline{r} = r'$ corresponds to a situation where the observation point falls on a source point, which frequently happens when setting up integrals equations for solutions of boundary value problems. We will exclude this point from the volume by surrounding it with a tiny sphere of radius a, as shown in fig. 8.5, and then take the limit as $a \to 0$. This is analogous to finding the principal value of an improper integral, which the reader may recall from advanced calculus.

Now substituting $\overline{A} = \overline{E}$ and $\psi = g$ into (8.56) and using (8.57) gives

$$\iiint_{V-V_0} \left[\left(\nabla \cdot \overline{E}\right) \nabla g - \left(\jmath\omega\mu\overline{J} + \nabla \times \overline{M}\right) g \right] dV$$
$$= \oiint_{S+S_0} \left[g\hat{n} \times \nabla \times \overline{E} + \left(\hat{n} \cdot \overline{E}\right) \nabla g + \left(\hat{n} \times \overline{E}\right) \times \nabla g \right] dS$$

From Maxwell's equations we have $\nabla \cdot \overline{E} = \rho_e/\epsilon$ and $\nabla \times \overline{E} = -\jmath\omega\mu\overline{H} - \overline{M}$. From (A.56) and (A.46) we can show that

$$\oiint d\overline{S} \times \left(\overline{M} g\right) = \iiint \nabla \times \left(\overline{M} g\right) \, dV = \iiint \left[g \nabla \times \overline{M} - \overline{M} \times \nabla g \right] dV$$

so Green's theorem leaves us with

$$\iiint_{V-V_0} \left[-\jmath\omega\mu\overline{J}g - \overline{M} \times \nabla g + \frac{\rho_e}{\epsilon}\nabla g \right] dV$$
$$= \oiint_{S+S_0} \left[-\jmath\omega\mu g \left(\hat{n} \times \overline{H}\right) + \left(\hat{n} \cdot \overline{E}\right)\nabla g + \left(\hat{n} \times \overline{E}\right) \times \nabla g \right] dS \quad (8.58)$$

This is in a convenient form, involving the volume sources on the left, and the surface fields on the right. Now lets consider the surface integral over S_0 (the exclusion surface) in the limit of $V_0 \to 0$. We will consider two cases: **(1)** when the observation point $\overline{r}'$ is completely within V, as depicted in fig. 8.5; and **(2)** $\overline{r}'$ lies on part of the surface S (shown later as fig. 8.7).

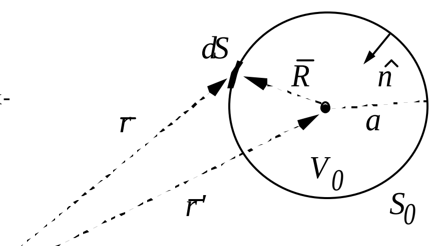


Figure 8.6 Expanded view of the exclusion volume of fig. 8.5.

An expanded view of the exclusion surface for the first case in shown in fig. 8.6. The particular solution for g in three-dimensions is (8.23); this is only part of the general solution for g which displays a singularity at $\overline{r} = \overline{r}'$. For convenience, g and ∇g are listed here as

$$g = \frac{e^{-\jmath kR}}{4\pi R} \qquad \nabla g = -\left(\frac{1 + jkR}{R}\right)\frac{e^{-\jmath kR}}{4\pi R}\hat{R} \quad (8.59)$$

Now, when integrating over S_0, the vector $\overline{r}$ will describe a point on the surface, so $\overline{R} = \overline{r} - \overline{r}'$ is a radial vector. If we take the S_0 to be a spherical surface of radius a, then $\overline{R} = a\hat{R} = -a\hat{n}$, where $\hat{n}$ is the unit normal to the surface, directed out of the volume V by convention. Since $dS \propto a^2$, then the first term on the right of (8.58) vanishes,

$$\lim_{a \to 0} \oiint_{S_0} \jmath\omega\mu g \left(\hat{n} \times \overline{H}\right) dS \to 0$$

Noting that $(\hat{n} \times \overline{E}) \times \nabla g = \overline{E}(\hat{n} \cdot \nabla g) - (\overline{E} \cdot \nabla g)\hat{n}$, the remaining part of the surface integral becomes

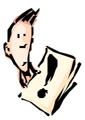
$$\lim_{a \to 0} \oiint_{S_0} \left[\left(\hat{n} \cdot \overline{E}\right)\hat{n} + \overline{E}\left(\hat{n} \cdot \hat{n}\right) - \left(\hat{n} \cdot \overline{E}\right)\hat{n} \right] \frac{e^{-jka}}{4\pi a}\frac{(1 + jka)}{a} a^2 \, d$$
$$\simeq \overline{E}(\overline{r}')\frac{1}{4\pi}\oiint_{S_0} d \quad = \overline{E}(\overline{r}') \quad (8.60)$$

Putting this back into our result (8.58) from Green's theorem, and using the common shortand

$$\iiint_V \equiv \lim_{V_0 \to 0} \iiint_{V-V_0}$$

gives the result

$$\begin{aligned}\overline{E}(\overline{r}') \; = \; & \iiint_V \left[-\jmath\omega\mu\overline{J}(\overline{r})g - \overline{M}(\overline{r}) \times \nabla g + \frac{\rho_e}{\epsilon}\nabla g\right] \, dV \\ & + \oiint_S \left[\jmath\omega\mu g \left(\hat{n} \times \overline{H}(\overline{r})\right) - \left(\hat{n} \cdot \overline{E}(\overline{r})\right) \nabla g - \left(\hat{n} \times \overline{E}(\overline{r})\right) \times \nabla g\right] \, dS\end{aligned} \tag{8.61}$$



This is often called the *Stratton-Chu solution* for the electric field (note that some books or literature use a different convention for the unit normal $\hat{n}$ and therefore have a different sign in front of the surface integral). Before we interpret it physically, lets see how this result changes in the second case where the obvservation point $\overline{r}'$ lies on a point of the surface S. Typically this is a point on a material boundary, as depicted in fig. 8.7. In

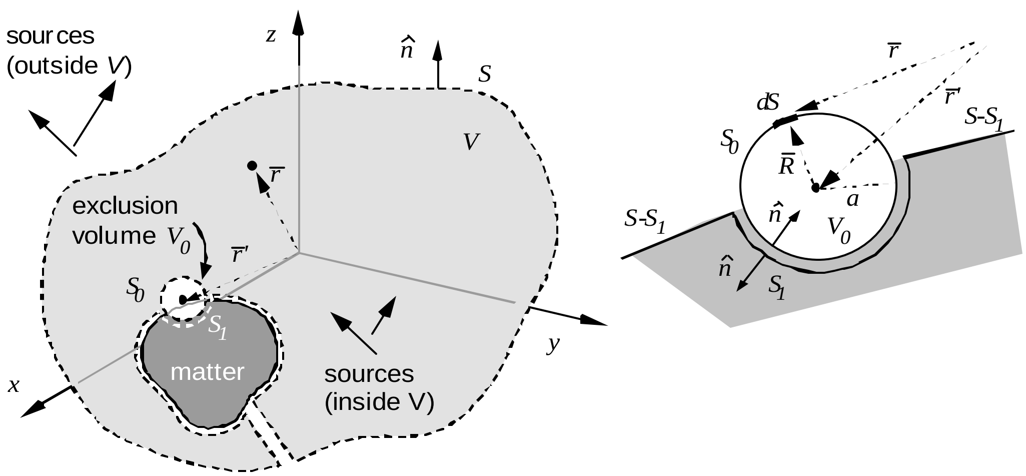


Figure 8.7 Same as fig. 8.5, but now the observation point lies on the surface S. The inset shows a closeup of this region.

this case the surface S must be deformed around the exclusion volume. We will call this part of the surface S_1 (fig. 8.7 again shows an expanded view of the exclusion region for clarity). So now the surface integral in (8.58) involves three parts

$$\oiint_{S+S_0} [\,] \, dS = \oiint_{S-S_1} + \oiint_{S_1} + \oiint_{S_0} [\,] \, dS$$

Clearly the integral over the surface S_0 will be unchanged from the previous case, giving the same result (8.60). The integral over S_1 is exactly the same as that over S_0, except that the unit normal points in the opposite direction, and the integration is over 2π steradians instead of 4π, and therefore

$$\lim_{a \to 0} \oiint_{S_1} [\,] \, dS \to -\frac{1}{2}\overline{E}(\overline{r}')$$

So when we put it all together,

$$\frac{1}{2}\overline{E}(\overline{r}') = \iiint_V \left[-\jmath\omega\mu\overline{J}(\overline{r})g - \overline{M}(\overline{r}) \times \nabla g + \frac{\rho_e}{\epsilon}\nabla g\right] dV + \oint\!\!\!\!\!\!\!\!-\!\!\oint_S \left[\jmath\omega\mu g\left(\hat{n} \times \overline{H}(\overline{r})\right) - \left(\hat{n} \cdot \overline{E}(\overline{r})\right)\nabla g - \left(\hat{n} \times \overline{E}(\overline{r})\right) \times \nabla g\right] dS \tag{8.62}$$

which is essentially the same form as (8.61), but differing by a factor of 2. The line through the surface integral symbol again reminds us to evaluate the integral carefully as a limiting process near the point $\overline{r} = \overline{r}'$.

We can represent both situations concisely as follows:

$$\tau\overline{E}(\overline{r}') = \iiint_V \left[-\jmath\omega\mu\overline{J}(\overline{r})g - \overline{M}(\overline{r}) \times \nabla g + \frac{\rho_e}{\epsilon}\nabla g\right] dV + \oiint_S \left[\jmath\omega\mu g\left(\hat{n} \times \overline{H}(\overline{r})\right) - \left(\hat{n} \cdot \overline{E}(\overline{r})\right)\nabla g - \left(\hat{n} \times \overline{E}(\overline{r})\right) \times \nabla g\right] dS \tag{8.63}$$

where

$$\tau = \begin{cases} 1 & \text{if } \overline{r}' \text{ in within } V \\ \frac{1}{2} & \text{if } \overline{r}' \text{ is on } S \end{cases}$$

We have also dropped the lines through the integral, with the understanding that *if we ever encounter the singular point in g during an integration, is should be evaluated using a limiting or "principal value" approach* as we have described.

We can go through a similar procedure to find an expression for the magnetic field as follows:

$$\tau\overline{H}(\overline{r}') = \iiint_V \left[-\jmath\omega\epsilon\overline{M}(\overline{r})g + \overline{J}(\overline{r}) \times \nabla g + \frac{\rho_m}{\mu}\nabla g\right] dV + \oiint_S \left[-\jmath\omega\epsilon g\left(\hat{n} \times \overline{E}(\overline{r})\right) - \left(\hat{n} \cdot \overline{H}(\overline{r})\right)\nabla g - \left(\hat{n} \times \overline{H}(\overline{r})\right) \times \nabla g\right] dS \tag{8.64}$$

We promised a physical interpretation of (8.63). The volume integral term clearly just represents that part of the field which is due to sources within the volume. The surface integral term is more interesting. It represents the part of the field due to sources *outside* of the volume, but in terms of fields evaluated on the surface S bounding the volume. This is connected with field equivalence principles, as we will discuss later. If the surface S everywhere coincided with a conducting boundary, then the surface integral term would represent the field in the volume due to the induced currents on the surface.

Note that our derivation made use of the free-space Green's function (8.23), *i.e.* the particular solution of (8.57b). Nevertheless, the results (8.63)-(8.64) apply for any general solution of (8.19). The reason is that a general solution for g will also include the complementary solutions of (8.19), but these will not be singular at $\overline{r} = \overline{r}'$ and hence will not contribute to the surface integral over S_0 as $a \to 0$.

8.4.2 Solution using Dyadic Green's Function

We can also derive a result in terms of the dyadic Green's function as follows. As shown in Problem 8.16, the volume integral term in (8.63) involving the charge density can be expressed as

$$\begin{aligned}\iiint_V \frac{\rho_e}{\epsilon}\nabla g\, dV \;=\; & \iiint_V \frac{1}{jwe}\left(\overline{J}\cdot\nabla\right)\nabla g\, dV \\ & + \oiint_S \left[\left(\hat{n}\cdot\overline{E}\right)\nabla g - \frac{1}{\jmath\omega\epsilon}\left(\hat{n}\times\overline{H}\right)\cdot\nabla\nabla g\right] dS\end{aligned} \tag{8.65}$$

Substituting this into (8.63), and recalling that the dyadic Green's function is related to the scalar Green's function as

$$\overline{\overline{G}}_e(\overline{r},\overline{r}') \equiv \left[\overline{\overline{I}} + \frac{\nabla\nabla}{k^2}\right] g(\overline{r},\overline{r}')$$

we can put (8.63) in the following form

$$\begin{aligned}\tau\overline{E}(\overline{r}') \;=\; & -\iiint_V \left[\jmath\omega\mu\,\overline{J}(\overline{r})\cdot\overline{\overline{G}}_e + \overline{M}(\overline{r})\cdot\nabla\times\overline{\overline{G}}_e\right] dV \\ & + \oiint_S \left[\jmath\omega\mu\left(\hat{n}\times\overline{H}(\overline{r})\right)\cdot\overline{\overline{G}}_e - \left(\hat{n}\times\overline{E}(\overline{r})\right)\cdot\nabla\times\overline{\overline{G}}_e\right] dS\end{aligned} \tag{8.66}$$

and, by duality,

$$\begin{aligned}\tau\overline{H}(\overline{r}') \;=\; & \iiint_V \left[\overline{J}(\overline{r})\cdot\nabla\times\overline{\overline{G}}_e - \jmath\omega\epsilon\,\overline{M}(\overline{r})\cdot\overline{\overline{G}}_e\right] dV \\ & - \oiint_S \left[\left(\hat{n}\times\overline{H}(\overline{r})\right)\cdot\nabla\times\overline{\overline{G}}_e + \jmath\omega\epsilon\,\left(\hat{n}\times\overline{E}(\overline{r})\right)\cdot\overline{\overline{G}}_e\right] dS\end{aligned} \tag{8.67}$$

As with the Stratton-Chu result, the integrals in (8.66)-(8.67) must be evaluated carefully near any singularities in the integrand. It is somewhat more difficult in this case because the dyadic Green's function is more strongly singular than the scalar Green's function, due to the $\nabla\nabla$ term. This is discussed in detail by Yanghjian [5].

8.4.3 Modified Green's Functions

Note that the scalar Green's function used in the Stratton-Chu solutions (8.63)-(8.64), and similarly the dyadic Green's function in (8.66)-(8.67), can be *any* solution of (8.57b) or (8.36), respectively. If the unbounded Green's function is used, then the solutions require a knowledge of *all* volume currents (impressed or induced), as well as both $\hat{n}\times\overline{E}$ and $\hat{n}\times\overline{H}$ on the bounding surface, which are generally not known simultaneously or *a priori*. However, there is a certain flexibility in the choice of the dyadic Green's function that can be exploited to simplify the problem. This flexibility arises from the ability to add any complementary solution of the homogeneous vector wave equation to a particular solution of (8.57b), as discussed earlier in connection with the scalar Green's functions. These complementary solutions allow us to modify the Green's function to satisfy specified boundary conditions.

For notational convenience we will focus the discussion on the dyadic Green's functions, which are always linked to the corresponding scalar Green's function using (8.35). The solutions we have found for the free-space dyadic Green's functions are particular solutions of the equations (8.36). We can construct more general Green's functions to satisfy prescribed boundary conditions by adding the complementary solutions $\overline{\overline{G}}_c$, which are solutions of the homogeneous equation

$$\nabla \times \nabla \times \overline{\overline{G}}_c + k^2\overline{\overline{G}}_c = 0$$

That is, we can express the general solutions as

$$\overline{\overline{G}}_e = \overline{\overline{G}}_{e0} + \overline{\overline{G}}_c$$

Note that the singularity of the modified Green's functions is contained entirely within the particular solutions. However, the symmetry relations (8.41) no longer hold *in general*.

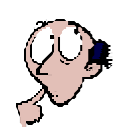
Let us now examine how the modified Green's function can help simplify (8.66). For convenience, we restrict attention to the case where the observation point is away from any volume sources, to avoid any unpleasantness with singularities. If, for example, the tangential electric field $\hat{n} \times \overline{E}(\overline{r}')$ is known on the boundary, we can eliminate the unknown term in the surface integral by constructing a *modified Green's function of the first kind* which vanishes on the boundary

$$\hat{n} \times \overline{\overline{G}}_{e1}(\overline{r}, \overline{r}') = 0 \qquad \text{for} \qquad \overline{r} \text{ on } S \tag{8.68}$$

since $(\hat{n} \times \overline{H}) \cdot \overline{\overline{G}} = -\overline{H} \cdot \hat{n} \times \overline{\overline{G}}$. For example, if the bounding surface is a PEC where $\hat{n} \times \overline{E} = 0$ on S, then choosing a modified Green's function of the first kind reduces (8.66) to

$$\overline{E}(\overline{r}') = -\iiint_V \left[\jmath\omega\mu\, \overline{J}(\overline{r}) \cdot \overline{\overline{G}}_{e1} + \overline{M}(\overline{r}) \cdot \nabla \times \overline{\overline{G}}_{e1} \right] dV \tag{8.69}$$

which is essentially the same form as was obtained for unbounded problems (8.38). Physically, this modified Green's function still represents the fields produced by a unit source, where the source is now radiating in the presence of material bodies. Therefore the modified Green's functions implicitly include the effects of induced currents on the conducting boundary. One way tounderstand this important point is to note that induced currents always arise so as to satisfy the boundary conditions, therefore any field solution that satisfies the boundary conditions must contain the response due to the induced currents.



Similarly, if $\hat{n} \times \overline{H}(\overline{r})$ in (8.66) is known, we can eliminate the unknown term in the surface integral by constructing a *modified Green's function of the second kind* which has a vanishing curl on the boundary

$$\hat{n} \times \nabla \times \overline{\overline{G}}_{e2}(\overline{r}, \overline{r}') = 0 \qquad \text{for} \qquad \overline{r} \text{ on } S \tag{8.70}$$

A careful discussion of the modified Green's functions can eb found in Tai [2]. In practice it is difficult to find dyadic Green's functions satisfying either condition above, with the exception of certain canonical geometries that are amenable to an eigenfunction expansion. For a general discussion of this method, see the book by Tai [2]. Spectral-domain techniques have also been developed to facilitate computation of the dyadic Green's functions in inhomogeneous or layered media [4].

8.5 SOMMERFELD RADIATION CONDITIONS

Now suppose all sources are contained with some finite region of space. If the bounding surface is enlarged to enclose all the sources then the surface integral in (8.63) should vanish, since it describes the field produced at $\overline{r}$ due to sources *external* to S. For any problem we can enclose all sources simply by letting the surface go to infinity, and so it must be that

$$\lim_{S\to\infty} \oiint_S \left[\jmath\omega\mu g \left(\hat{n}\times\overline{H}\right) - \left(\hat{n}\cdot\overline{E}\right)\nabla g - \left(\hat{n}\times\overline{E}\right)\times\nabla g\right] dS \Rightarrow 0 \qquad (8.71)$$

This leads to the so-called "radiation conditions" on the field, which are required to insure a vanishing surface integral at infinity. If we take the surface to be a sphere of infinite radius centered at $\overline{r}$, then the outward normal $\hat{n}$ at the point $\overline{r}'$ on the surface is just $\hat{n} = -\hat{R}$, since $\hat{R}$ points towards $\overline{r}$. Using (8.59), and $dS = R^2 d$, the surface integral (8.71) becomes

$$\lim_{R\to\infty} \oiint_S \Bigg\{ -\jmath\omega\mu g \left(\hat{R}\times\overline{H}\right) + \left(\frac{1}{R} + \jmath k\right) \underbrace{\left[\left(\hat{R}\cdot\overline{E}\right)\hat{R} - \hat{R}\times\left(\hat{R}\times\overline{E}\right)\right]}_{\overline{E}} \Bigg\} \frac{e^{-\jmath kR}}{4\pi R} R^2\, d \quad = 0$$

which can be written as

$$\lim_{R\to\infty} \oiint_S \left[-\jmath kR\left(\eta\hat{R}\times\overline{H} + \overline{E}\right) + \overline{E}\right] \frac{e^{-\jmath kR}}{4\pi}\, d \quad = 0$$

where $\eta = \sqrt{\mu/\epsilon}$. Similarly, if we had started by trying to make the surface integral vanish in the solution for the magnetic field (8.64), we would have found

$$\lim_{R\to\infty} \oiint_S \left[-\jmath kR\left(\overline{H} - \frac{1}{\eta}\hat{R}\times\overline{E}\right) + \overline{H}\right] \frac{e^{-\jmath kR}}{4\pi}\, d \quad = 0$$

In order for these last two integrals to vanish, the fields must obey the following conditions:

a) $\left\{\begin{matrix}\overline{E}\\ \overline{H}\end{matrix}\right\}$ drop off at least as fast as $1/R$ (8.72a)

b) $\lim_{R\to\infty} R\left[\overline{H} - \hat{R}\times\frac{\overline{E}}{\eta}\right] = 0$ or $\lim_{R\to\infty} R\left[\frac{\overline{E}}{\eta} + \hat{R}\times\overline{H}\right]$ (8.72b)

The first conditions make the last term in the integrals (inside the square brackets) vanish in the limit $R \to \infty$. The last conditions are called the *Sommerfeld radiation conditions* and imply that the fields at infinity are outward-propagating spherical TEM waves, as expected on physical grounds. Together these constraints provide the necessary boundary conditions to obtain a unique solution to Maxwell's equations in unbounded space. So, fields produced by some localized source are always expected to have the form

$$\left\{ \frac{\overline{E}}{\overline{H}} \right\} \propto \frac{e^{-jkr}}{r} \overline{f}(\theta, \phi) \tag{8.73}$$

where $\overline{f}$ is a vector that describes the angular dependence and polarization of the radiation, and arises in various contexts under such names as the *scattering amplitude*, *Fraunhoffer diffraction pattern*, and *far-field radiation pattern*.

8.6 HUYGEN'S PRINCIPLE

Huygen's principle states that the field at any point in space is a superposition of spherical waves emanating from a surface enclosing all the sources. There are several equivalent mathematical representations of Huygen's principle that are important in radiation theory. We will derive a general result in terms of the dyadic Green's function, and show that various choices for the Green's function result in formulations of the field equivalence principles discussed in Chapter 1.

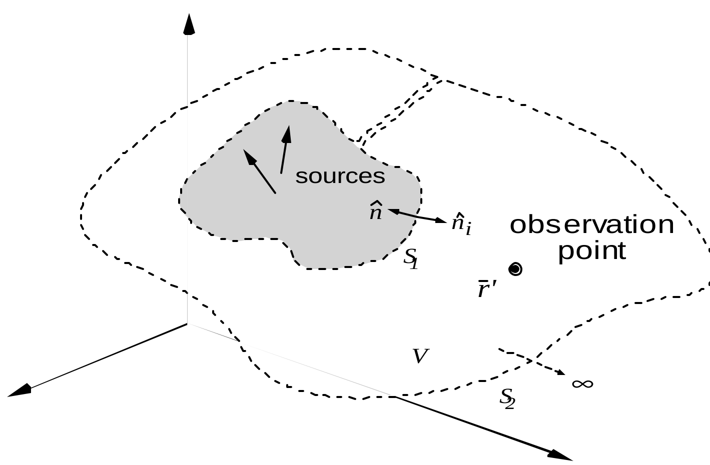
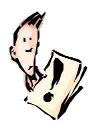


Figure 8.8 Illustration showing exclusion of sources from a volume V for derivation of Huygen's integral.

Consider the problem shown in fig. 8.8, which shows a set of localized sources. We are interested in the fields produced external to these sources. We can define a volume V which excludes these currents using two simply-connected surfaces S_1 and S_2. If we let the surface S_2 go to infinity, then the integral over S_2 will vanish by virtue of the radiation conditions. Applying our dyadic Green's function formalism (8.66) and (8.67), the fields at some observation point $\overline{r}'$ in the volume are given by



$$\tau\overline{E}(\overline{r}') = \oiint_{S_1} \left[-\jmath\omega\mu\left(\hat{n}_i \times \overline{H}(\overline{r})\right)\cdot\overline{\overline{G}}_e + \left(\hat{n}_i \times \overline{E}(\overline{r})\right)\cdot\nabla\times\overline{\overline{G}}_e\right] dS \qquad (8.74a)$$

$$\tau\overline{H}(\overline{r}') = \oiint_{S_1} \left[\left(\hat{n}_i \times \overline{H}(\overline{r})\right)\cdot\nabla\times\overline{\overline{G}}_e + \jmath\omega\epsilon\left(\hat{n}_i \times \overline{E}(\overline{r})\right)\cdot\overline{\overline{G}}_e\right] dS \qquad (8.74b)$$

where we have defined the integrand in terms of $\hat{n}_i = -\hat{n}$, which points *out* of the region enclosed by S_1 and hence *into* the volume V. These equations are a formal statement of Huygen's principle, and are the basis for much of diffraction and scattering theory.

8.6.1 Field Equivalence Revisited

The above statement of Huygen's principle is intimately connected with the field equivalence principles, which can be seen by making the following substitutions

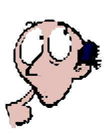
$$\hat{n}_i \times \overline{H}(\overline{r}) = \overline{J}_s(\overline{r})$$
$$\hat{n}_i \times \overline{E}(\overline{r}) = -\overline{M}_s(\overline{r})$$

which gives

$$\tau\overline{E}(\overline{r}') = -\oiint_{S_1} \left[\jmath\omega\mu\overline{J}_s(\overline{r})\cdot\overline{\overline{G}}_e + \overline{M}_s(\overline{r})\cdot\nabla\times\overline{\overline{G}}_e\right] dS \qquad (8.75a)$$

$$\tau\overline{H}(\overline{r}') = \oiint_{S_1} \left[\overline{J}_s(\overline{r})\cdot\nabla\times\overline{\overline{G}}_e - \jmath\omega\epsilon\overline{M}_s(\overline{r})\cdot\overline{\overline{G}}_e\right] dS \qquad (8.75b)$$

The currents $\overline{J}_s$ and $\overline{M}_s$ are *equivalent impressed* surface currents, and only relate to actual currents if the surface S_1 coincides with a conducting boundary. These field expressions are only valid inside the source-free volume V; the regions outside of the volume of interest have been effectively accounted for using equivalent surface currents flowing on the bounding surface S_1. In other words, the region external to V is irrelevant to the determination of the fields inside V, and can be effectively replaced by any medium that proves convenient. If the original source region is replaced by the same material occupying V, then we have effectively replaced the original source distribution by a set of impressed currents flowing on S_1 and radiating into an unbounded medium. In this case, (8.75) is a mathematical statement of Love's equivalence principle. The unbounded dyadic Green's function is therefore suitable for such problems.



For example, suppose we have a conducting rod in an unbounded region with a *known total* current distribution $\overline{J}_s$ flowing on it, as shown in fig. 8.9. These currents may be induced currents due to an incident plane wave, for example. According to Huygen's principle, the fields radiated by this current can be expressed as

$$\tau\overline{E}(\overline{r}) = -\jmath\omega\mu\oiint_{S} \overline{\overline{G}}_{e0}(\overline{r},\overline{r}')\cdot\overline{J}_s(\overline{r}')\,dS' \qquad (8.76)$$

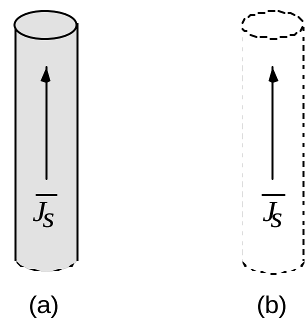


Figure 8.9 (a) Conducting rod carrying known current $\overline{J}_s$. (b) Equivalent for calculation of external fields.

where $\overline{\overline{G}}_{e0}$ is the unbounded dyadic Green's function, and we have made use of the reciprocal property (8.41). This is just the expression we would get from (8.38) if we considered the surface current distribution to be radiating in free-space without the conducting rod present. *For the purpose of calculating the external fields, the interior region is irrelevant* and can be effectively replaced by free-space. This result is often invoked implicitly in antenna analysis, and is just the Love's equivalent discussed earlier.

A different example illustrates how the statement of Huygen's principle (8.74) can be interpreted in terms of the Schelkunoff equivalents, using modified dyadic Green's functions. Consider the situation in fig. 8.10a, where a set of sources radiates through an aperture in a conducting boundary. We assume that the aperture fields are known or can be approximated in some way. Huygen's principle allows us to express the fields external to the object in terms of the tangential fields on the surface S. Using the ideas described above gives the Love's equivalent of fig. 8.10b, where the object and the original sources are replaced by the equivalent currents as shown. The equivalent impressed currents in this case consist of the original induced currents flowing on the external surface of the conductor ($\overline{J}_f$), plus the equivalent currents $(\overline{J}_s, \overline{M}_s)$ due to the aperture fields. The fields can then be computed from (8.75) using the unbounded Green's function $\overline{\overline{G}}_{e0}$.

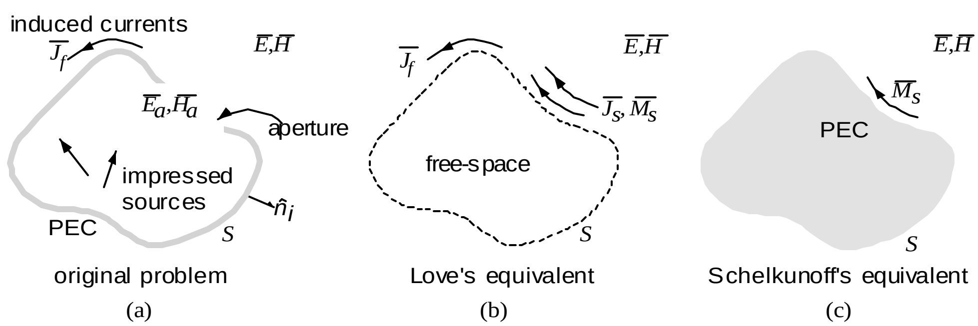


Figure 8.10 Aperture diffraction problem and equivalents.

A difficulty with Love's equivalent in fig. 8.10b is that the induced currents $\overline{J}_f$ are not known *a priori*; these currents are induced by the sources within the shell (or equivalently the aperture fields), and may be difficult or unwieldy to determine for large or complicated surfaces. However, the term in the surface integral of (8.75) involving $\overline{J}_f$ can be eliminated by constructing a modified Green's function of the first kind that satisfies

$$\hat{n}_i \times \overline{\overline{G}}_{e1}(\overline{r}, \overline{r}') = 0 \qquad \text{on} \quad S$$

in which case

$$\tau \overline{E}(\overline{r}') \;=\; -\oiint_S \overline{M}_s(\overline{r}) \cdot \nabla \times \overline{\overline{G}}_{e1}(\overline{r}, \overline{r}')\, dS \tag{8.77a}$$

$$\tau \overline{H}(\overline{r}') \;=\; -\jmath\omega\epsilon \oiint_S \overline{M}_s(\overline{r}) \cdot \overline{\overline{G}}_{e1}(\overline{r}, \overline{r}')\, dS \tag{8.77b}$$

This corresponds to placing a PEC shell on the original surface, which gives the Schelkunoff equivalent shown in fig. 8.10c. In this case only the impressed magnetic currents over the surface near the original aperture are required. Physically there will still be currents induced on the PEC surface by the impressed magnetic current, but these are accounted for implicitly by the modified Green's function. Note that this beautiful result comes at a cost: one must now find a modified Green's function which satisfies the boundary conditions on the object! This is usually as difficult as the original problem of determining the induced currents on the boundary in the Love's equivalent (if not more difficult!). Fortunately, a large volume of research effort has gone into developing analytical techniques for determining or approximating such Green's functions for many problems of practical interest. We will make use of some of this work in later chapters. Therefore, we conclude that (8.77) has significant merit as a concise symbolic representation of the field, and possibly for computational work provided that an appropriate Green's function can be found. The primary reason for discussing the result here is to better illustrate the ideas behind the field equivalence principles.

For completeness, we should also present the Schelkunoff equivalent using a PMC surface. This might arise in situations where the magnetic equivalent currents are difficult to find or approximate, in which case they can be eliminated from (8.75) by choosing a modified Green's function of the second kind that satisfies

$$\hat{n}_i \times \left(\nabla \times \overline{\overline{G}}_{e2}(\overline{r}, \overline{r}')\right) = 0 \qquad \text{on} \quad S$$

in which case the fields are then given by

$$\tau \overline{E}(\overline{r}') \;=\; -\jmath\omega\mu \oiint_S \overline{J}_s(\overline{r}) \cdot \overline{\overline{G}}_{e2}(\overline{r}, \overline{r}')\, dS \tag{8.78a}$$

$$\tau \overline{H}(\overline{r}') \;=\; \oiint_S \overline{J}_s(\overline{r}) \cdot \nabla \times \overline{\overline{G}}_{e2}(\overline{r}, \overline{r}')\, dS \tag{8.78b}$$

8.6.2 Stratton-Chu Formulation

Huygen's principle can of course be expressed in terms of the scalar Green's function using (8.63)-(8.64) as follows,

$$\tau \overline{E}(\overline{r}) \;=\; -\oiint_{S_1} \left[\jmath\omega\mu \left(\hat{n}_i \times \overline{H}(\overline{r}')\right) g\right.$$

$$+ \ \left(\hat{n}_i \cdot \overline{E}(\overline{r}')\right) \nabla g + \nabla g \times \left(\overline{E}(\overline{r}') \times \hat{n}_i\right)\big] \ dS'$$

and by duality,

$$\begin{aligned} \tau \overline{H}(r) &= -\oiint_{S_1} \left[\jmath\omega\epsilon \left(\overline{E}(\overline{r}') \times \hat{n}_i\right) g \right. \\ &\quad + \left[\left(\hat{n}_i \cdot \overline{H}(\overline{r}')\right) \nabla g - \nabla g \times \left(\hat{n}_i \times \overline{H}(\overline{r}')\right)\right] \ dS' \end{aligned}$$

where we have interchanged $\overline{r}$ and $\overline{r}'$ so that the integration is now over primed coordinated. Note also the negative sign in front of the surface integrals; this has changed due to the use of $\hat{n}_i$. These represent the *Stratton-Chu* formulation of Huygen's principle. Note that both normal and tangential components of the fields are required for this formulation. It is known from the uniqueness theorem that a field within a bounded region is uniquely determined if the tangential fields alone are specified on the boundary. Therefore $\hat{n}_i \cdot \overline{E}$ and $\hat{n}_i \cdot \overline{H}$ must be chosen consistent with the tangential fields through Maxwell's equation.

8.6.3 Franz Relations

The Stratton-Chu results can be cast in a form involving only tangential components of the electric field. The offending terms in (8.78) and (8.78) involving the normal components of the fields are eliminated by taking the curl of both sides of the equations, since these terms are proportional to ∇g. For example,

$$\begin{aligned} \nabla \times \overline{H}(\overline{r}) &= -\jmath\omega\epsilon \nabla \times \oiint_{S_1} \left(\overline{E}(\overline{r}') \times \hat{n}_i\right) g \, dS' \\ &\quad - \nabla \times \nabla \times \oiint_{S_1} \left(\hat{n}_i \times \overline{H}(\overline{r}')\right) g \, dS' \end{aligned}$$

where we have used the fact that $\nabla g \times (\hat{n}_i \times \overline{H}(\overline{r}')) = \nabla \times \left[(\hat{n}_i \times \overline{H}(\overline{r}'))g\right]$. Using Maxwell's equation $\nabla \times \overline{H} = \jmath\omega\epsilon\overline{E}$, and following the same procedure with the equation for $\overline{E}$ gives

$$\begin{aligned} \tau \overline{E}(\overline{r}) &= -\frac{1}{\jmath\omega\epsilon} \nabla \times \nabla \times \oiint_{S_1} \hat{n}_i \times \overline{H}(\overline{r}') g \, dS' \qquad (8.79a) \\ &\quad + \nabla \times \oiint_{S_1} \hat{n}_i \times \overline{E}(\overline{r}') g \, dS' \\ \tau \overline{H}(\overline{r}) &= -\nabla \times \oiint_{S_1} \hat{n}_i \times \overline{H}(\overline{r}') g \, dS' \qquad (8.79b) \\ &\quad - \frac{1}{\jmath\omega\mu} \nabla \times \nabla \times \oiint_{S_1} \hat{n}_i \times \overline{E}(\overline{r}') g \, dS' \end{aligned}$$

These are the Franz formulas, and are reminiscent of the Hertz vector results, where we found that in source free regions the fields can be expressed as

$$\begin{aligned} \tau \overline{E}(\overline{r}) &= \nabla \times \nabla \times \overline{\Pi}_e(\overline{r}) - \jmath\omega\mu \nabla \times \overline{\Pi}_m(\overline{r}) \\ \tau \overline{H}(\overline{r}) &= \jmath\omega\epsilon \nabla \times \overline{\Pi}_e(\overline{r}) + \nabla \times \nabla \times \overline{\Pi}_m(\overline{r}) \end{aligned}$$

So evidently

$$\overline{\Pi}_e(\overline{r}) = -\frac{1}{\jmath\omega\epsilon}\oiint_{S_1} \hat{n}_i \times \overline{H}(\overline{r}')g(\overline{r},\overline{r}')\,dS' \tag{8.80a}$$

$$\overline{\Pi}_m(\overline{r}) = -\frac{1}{\jmath\omega\mu}\oiint_{S_1} \hat{n}_i \times \overline{E}(\overline{r}')g(\overline{r},\overline{r}')\,dS' \tag{8.80b}$$

In fact, these are exactly the expressions obtained directly from a solutions of the equations for the Hertz potentials in unbounded homogeneous regions, with the volume currents replaced by surface currents; this is another expression of Love's equivalence.

8.7 FAR-FIELD SOLUTIONS

8.7.1 Far-fields in 3D

We now focus on possible simplifications for arbitrarily large source distributions when only the radiation fields (varying as $1/r$) are of interest. We start by making a Taylor

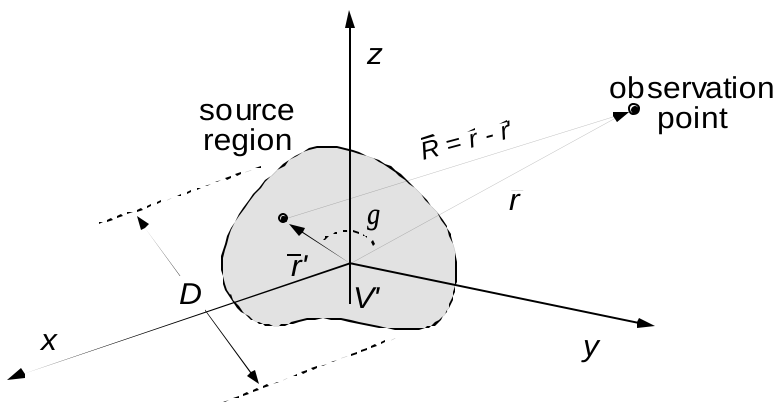


Figure 8.11 Source region with a characteristic dimension D for delimiting far-field region.

expansion of $R = |\overline{r} - \overline{r}'|$ for the case of $|\overline{r}| \gg |\overline{r}'|$, giving

$$\begin{aligned} R = |\overline{r} - \overline{r}'| &= [(\overline{r} - \overline{r}')\cdot(\overline{r} - \overline{r}')]^{\frac{1}{2}} = \left[r^2 - 2\overline{r}\cdot\overline{r}' + r'^2\right]^{\frac{1}{2}} \\ &= \left[r^2\left(1 - \frac{2\overline{r}\cdot\overline{r}'}{r^2} + \frac{r'^2}{r^2}\right)\right]^{\frac{1}{2}} \\ &= r\left[1 - \frac{\overline{r}\cdot\overline{r}'}{r^2} + \frac{1}{2}\frac{r'^2}{r^2} - \frac{1}{2}\frac{(\overline{r}\cdot\overline{r}')^2}{r^4} + \ldots\right] \end{aligned} \tag{8.81}$$

Denoting the angle between $\overline{r}$ and $\overline{r}'$ as γ (see figure 8.11) we can write $\overline{r}\cdot\overline{r}' = rr'\cos\gamma$ where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') \tag{8.82}$$

and so the first three terms of the expansion in R become

$$R \approx r - \hat{r}\cdot\overline{r}' + \frac{1}{2}\frac{r'^2}{r}\sin^2\gamma + \ldots$$

where $\hat{r} = \overline{r}/r$. When substituted into the exponential term of the Green's function e^{-jkR} we must keep at least the first two terms, since though $\overline{r}'$ is small compared with $\overline{r}$ it can still be large relative to a wavelength, and consequently kr' can represent a non-negligible phase. The third term contributes a phase of

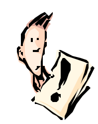
$$\frac{1}{2}\frac{kr'^2}{r}\sin^2\gamma$$

We can always choose r far enough away so that this term is negligible. A useful—though somewhat arbitrary—rule-of-thumb defines "negligible" to mean a phase contribution of $\pi/8$ (22.5°) or less. Then, if the source distribution is centered on the origin and has a characteristic length of D so that $|r'| \leq D/2$, the critical observation distance should be such that

$$\frac{1}{2}\frac{k\left(\frac{D}{2}\right)^2}{r} < \frac{\pi}{8}$$

or

$$r > \frac{2D^2}{\lambda} \qquad \text{(condition for far-field)} \tag{8.83}$$



When this condition holds we are in the "far-field" of the source distribution. Then

$$R \approx r - \hat{r}\cdot\overline{r}' \qquad \text{(far-field approximation)} \tag{8.84}$$

and the Green's function becomes

$$\frac{e^{-jkR}}{R} \approx e^{-jkr}e^{jk\hat{r}\cdot\overline{r}'}\frac{1}{r}\left[1 + \frac{\hat{r}\cdot\overline{r}'}{r} + \ldots\right]$$

The radiation fields (varying as $1/r$) become the dominant term at a distance of $r > D/2$. This condition is automatically satisfied in the far-field defined by (8.83), unless the antenna is very small compared with a wavelength ($D < \lambda/4$), in which case the expansion (8.42) would be more appropriate. In any case, we can always choose a distance far enough away from the antenna so that only the radiation fields are important, so the far-field Green's function is

$$g_{ff}(\overline{r}, \overline{r}') \Rightarrow \frac{e^{-jkr}}{4\pi r}e^{jk\hat{r}\cdot\overline{r}'} \tag{8.85}$$

and our far-field vector potentials are therefore

$$\overline{A}(\overline{r}) \;\Rightarrow\; \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\iiint \overline{J}(\overline{r}')e^{+jk\hat{r}\cdot\overline{r}'}\,dV' \tag{8.86a}$$

$$\overline{F}(\overline{r}) \;\Rightarrow\; \frac{\epsilon}{4\pi}\frac{e^{-jkr}}{r}\iiint \overline{M}(\overline{r}')e^{jk\hat{r}\cdot\overline{r}'}\,dV' \tag{8.86b}$$

Note that the integrand of these "radiation integrals" does not depend on the observation distance r. Now consider the calculation of the fields from (8.13). In spherical coordinates

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Only the $\hat{r}$ component of the ∇ operator is capable of generating fields with $1/r$ dependence; the other components produce terms going as $1/r^2$, which we have so far consistently neglected. Keeping only these terms in $1/r$ gives

$$\overline{A} + \frac{1}{k^2}\nabla\nabla\cdot\overline{A} \Rightarrow \overline{A} - (\hat{r}\cdot\overline{A})\hat{r} = A_\theta\hat{\theta} + A_\phi\hat{\phi}$$
$$\nabla\times\overline{F} \Rightarrow -\jmath k(\hat{r}\times\overline{F}) = \jmath k(F_\phi\hat{\theta} - F_\theta\hat{\phi})$$

so from (8.13) the radiation fields are

$$\overline{E}(\overline{r}) = -\jmath\omega\left[A_\theta\hat{\theta} + A_\phi\hat{\phi}\right] - \frac{\jmath k}{\epsilon}\left[F_\phi\hat{\theta} - F_\theta\hat{\phi}\right] \qquad (8.87a)$$

$$\overline{H}(\overline{r}) = \frac{\hat{r}\times\overline{E}}{\eta} \qquad (8.87b)$$

Writing these out explicitly for later use, we have, for electric sources only,

$$\overline{E}(\overline{r}) = -\jmath\omega\mu\frac{e^{-\jmath kr}}{4\pi r}\iiint\left[\overline{J}(\overline{r}') - \left(\hat{r}\cdot\overline{J}(\overline{r}')\right)\hat{r}\right]e^{+\jmath k\hat{r}\cdot\overline{r}'}\,dV' \qquad (8.88a)$$

$$\overline{H}(\overline{r}) = -\jmath k\frac{e^{-\jmath kr}}{4\pi r}\iiint\hat{r}\times\overline{J}(\overline{r}')\,e^{+\jmath k\hat{r}\cdot\overline{r}'}\,dV' \qquad (8.88b)$$

and for magnetic sources only,

$$\overline{E}(\overline{r}) = \jmath k\frac{e^{-\jmath kr}}{4\pi r}\iiint\hat{r}\times\overline{M}(\overline{r}')\,e^{+\jmath k\hat{r}\cdot\overline{r}'}\,dV' \qquad (8.89a)$$

$$\overline{H}(\overline{r}) = -\jmath\omega\epsilon\frac{e^{-\jmath kr}}{4\pi r}\iiint\left[\overline{M}(\overline{r}') - \left(\hat{r}\cdot\overline{M}(\overline{r}')\right)\hat{r}\right]e^{+\jmath k\hat{r}\cdot\overline{r}'}\,dV' \qquad (8.89b)$$

These are the so-called "radiation equations." It is clear from these equations that for any particular observation point $\overline{r}$, the far-field has the form of a spherical TEM wave propagating in the $\hat{r}$ direction, and is due to only that part of the current distribution that is flowing transverse to $\hat{r}$.

Far-Fields in Dyadic Notation

Substituting (8.85) into (8.35) we find, in the far-field,

$$\overline{\overline{G}}_{e0}(\overline{r},\overline{r}') \Rightarrow (\overline{\overline{I}} - \hat{r}\hat{r})\frac{e^{-\jmath kr}}{4\pi r}e^{\jmath k\overline{r}'\cdot\hat{r}} \qquad (8.90a)$$

$$\nabla\times\overline{\overline{G}}_{e0}(\overline{r},\overline{r}') \Rightarrow -\jmath k(\hat{r}\times\overline{\overline{I}})\frac{e^{-\jmath kr}}{4\pi r}e^{\jmath k\overline{r}'\cdot\hat{r}} \qquad (8.90b)$$

and so (8.39) reduce to, for electric currents,

$$\overline{E}(\overline{r}) = -\jmath\omega\mu\frac{e^{-\jmath kr}}{4\pi r}(\overline{\overline{I}} - \hat{r}\hat{r}) \cdot \iiint \overline{J}(\overline{r}')e^{\jmath k\overline{r}'\cdot\hat{r}}\, dV \tag{8.91a}$$

$$\overline{H}(\overline{r}) = -\jmath k\frac{e^{-\jmath kr}}{4\pi r}\hat{r} \times \iiint \overline{J}(\overline{r}')e^{\jmath k\overline{r}'\cdot\hat{r}}\, dV \tag{8.91b}$$

which are, of course, identical to the previously derived result (8.88), but using dyadic notation. A similar result for magnetic currents can be written down by inspection of (8.89).

8.7.2 Far-fields in 2D

For problems in which variation of fields is negligible in one cartesian direction (taken as the $\hat{z}$-axis) then the vector potential is given by (8.29a),

$$\overline{A}(\overline{\rho}) = \frac{\mu}{4j}\iint \overline{J}(\overline{\rho}')H_0^{(2)}(kR)\, dS'$$

where

$$\begin{aligned} R = |\overline{\rho} - \overline{\rho}'| &= \sqrt{(x-x')^2 + (y-y')^2} \\ &= \sqrt{\rho^2 + {\rho'}^2 - 2\rho\rho'\cos(\phi - \phi')} \end{aligned}$$

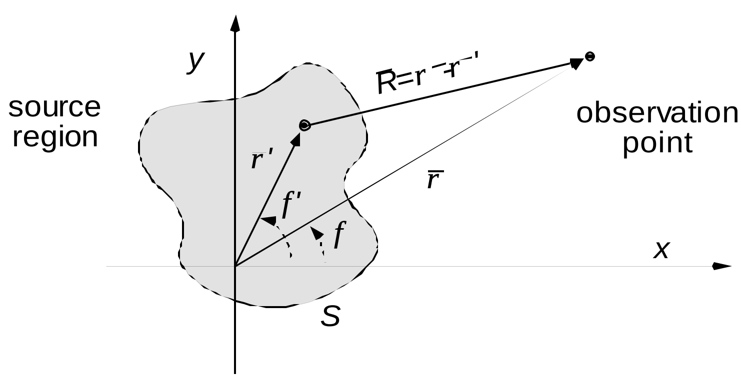


Figure 8.12 Two-dimensional source distribution (infinite extent in the $\hat{z}$ direction).

In the far-field, $kR \gg 1$, and we can use the asymptotic expansion for the 2D Green's function from (8.27),

$$H_0^{(2)}(kR) \approx \sqrt{\frac{2}{\pi kR}}e^{-j(kR-\frac{\pi}{4})}$$

For $|\overline{\rho}| \gg |\overline{\rho}'|$, then we can make a Taylor expansion of R much like for the 3D case, giving

$$\begin{aligned} R &\cong \rho - \hat{\rho}\cdot\overline{\rho}' \\ &\cong \rho - \overline{\rho}'\cos(\phi - \phi') \end{aligned} \tag{8.92}$$

or we could express R as

$$R \cong \rho - x'\cos\phi - y'\sin\phi$$

Keeping only terms in $1/\sqrt{\rho}$ (radiation fields in 2D) gives

$$\overline{A}(\overline{\rho}) = \frac{\mu}{2}\frac{e^{-jk\rho}}{\sqrt{2jk\rho\pi}}\iint_S \overline{J}(\overline{\rho}')e^{jk\hat{\rho}\cdot\overline{\rho}'}\,dS' \tag{8.93}$$

and the electric field in the far-field is given by

$$\overline{E}(\overline{\rho}) \cong -j\omega(A_\phi\hat{\phi} + A_z\hat{z}) \tag{8.94}$$

The extension to magnetic currents is straightforward. It is very important to remember when computing A_ϕ that the $\hat{\phi}$ direction at the source and observation points are general-lydifferent and must be handled accordingly. If the current is best described by rectangular components, then

$$A_\phi = \frac{\mu}{2}\frac{e^{-jk\rho}}{\sqrt{2jk\rho\pi}}\iint_S [-J_x(\overline{\rho}')\sin\phi + J_y(\overline{\rho}')\cos\phi]\,e^{jk\hat{\rho}\cdot\overline{\rho}'}\,dS' \tag{8.95a}$$

$$A_z = \frac{\mu}{2}\frac{e^{-jk\rho}}{\sqrt{2jk\rho\pi}}\iint_S J_z(\overline{\rho}')e^{jk\hat{\rho}\cdot\overline{\rho}'}\,dS' \tag{8.95b}$$

Note that the explicit $\sin\phi$ and $\cos\phi$ terms in (8.95a) are functions of the *unprimed* ϕ (at observation point). If $\overline{J}$ is best described in cylindrical coordinates then we can again use (8.95) but with the substitutions

$$J_x = J_\rho\cos\phi' - J_\phi\sin\phi' \tag{8.96a}$$

$$J_y = J_\rho\sin\phi' + J_\phi\cos\phi' \tag{8.96b}$$

where these terms involve the primed (source) variable ϕ'.

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