

10.5 ANALYSIS OF JUNCTIONS AND DISCONTINUITIES

For a given waveguide cross section the fields of a closed cylindrical waveguide can be described by an infinite set of TE and TM modes. Only a finite number of these modes can propagate at a given frequency; the rest are evanescent.

Typical problems involving waveguides include the scattering of waves from obstacles placed in the waveguide (figure 10.7), junctions between dissimilar waveguides, or excitation of waveguides by sources placed within them. The obstacles or sources introduce new boundary conditions, so that the original TE and TM modes of the empty waveguide are no longer satisfactory solutions for the fields near the obstacle. However, since the empty-waveguide modes form a complete orthogonal set, any field can be represented as a superposition of the modes. Note that this representation must theoretically include all the modes, whether evanescent or propagating. These empty-waveguide modes are an especially convenient basis since they already satisfy the boundary conditions on the waveguide walls.

After representing the fields near the obstacle or junction in this way, the expansion coefficients are then found by enforcing the boundary conditions imposed by the source or obstacle, and exploiting the mode orthogonality property. The procedure is directly analogous to Fourier series analysis; in the context of waveguide junctions or scattering problems, it is usually called “mode-matching”.

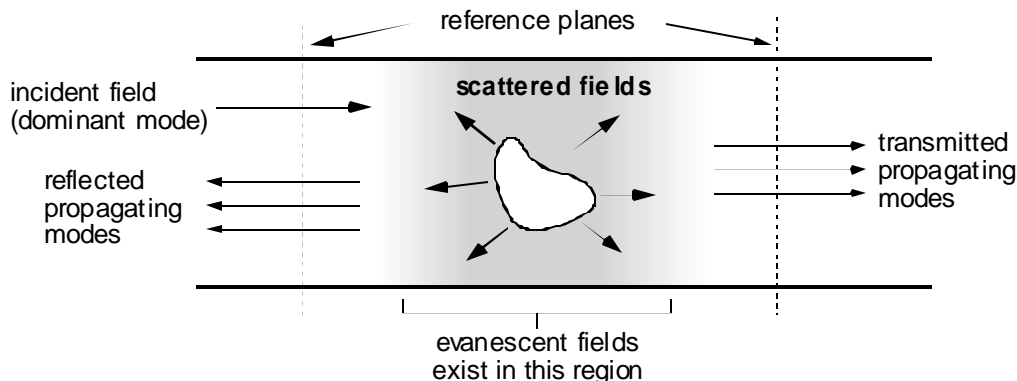


Figure 10.7 Typical waveguide scattering problem. The scattered fields can be represented as a superposition of propagating and evanescent modes, as dictated by the boundary conditions. At suitably chosen reference places far from the obstacle, only propagating modes are present, and the system can be modeled as a N -port network, where N is the number of propagating modes.

The goal of analysis for problems such as in figure 10.7 is usually to determine the effect of the obstacle on the propagating modes of the system. By defining a set of reference planes far enough from the obstacle so that all the evanescent fields have decayed to negligible amplitude, the obstacle can be characterized entirely by the amplitudes of the propagating modes. In many cases, only the dominant mode is allowed to propagate, in which case the region within the reference planes can be characterized by two complex numbers, the

reflection coefficient and transmission coefficient. This is similar to an ideal transmission-line junction. We will find that, in general, complicated waveguide problems can often be characterized by equivalent N -port circuits.

The detailed nature of the fields near the obstacle are only important to the extent that they influence the amplitudes of the propagating modes. We will find that the expressions for the equivalent circuit elements for obstacles or junctions are often relatively insensitive to errors in the near-fields of the junction, and hence reasonably accurate results can be obtained with some educated guesswork about the fields.

10.5.1 Analysis of Planar Discontinuities by Mode-Matching

Simple discontinuity or waveguide junction problems that involve planar boundaries (discontinuities lying in a $z=\text{constant}$ plane) can be treated in a straightforward manner by field-matching techniques. As an illustration of this method we will consider the waveguide junction shown in figure 10.8. The true fields on either side of the junction can be



Figure 10.8 Simple waveguide junction problem involving a reduction in cross section from S_a to S_b at $z = 0$ and a PEC boundaries.

represented by a modal expansion. The unknown coefficients are then found by enforcing the continuity of tangential fields across the junction and exploiting mode orthogonality. This procedure results in a matrix equation for the unknown coefficients, which is solved numerically. Strictly speaking the method is rigorous only if an infinite number of modes are considered. However, excellent results can usually be obtained by truncating the expansions to a finite number of terms. This is true because any high-order evanescent modes excited near the junction contribute little to the terms involving the dominant modes if their cutoff frequencies are well above the operating frequency. The number of terms required to obtain adequate results is very problem-specific, and usually must be determined empirically.

We assume that the junction is excited from the left by the dominant mode of unit amplitude in waveguide a . Just before the junction on the left ($z = 0^-$) we write

$$\overline{E}_t(z = 0^-) = \hat{e}_{a1} + \sum_{i=1}^{N_a} a_i \hat{e}_{ai} \quad (10.61a)$$

$$\overline{H}_t(z = 0^-) = \hat{h}_{a1} - \sum_{i=1}^{N_a} a_i \hat{h}_{ai} \quad (10.61b)$$

and to the right at $z = 0^+$ we can express the transmitted fields in terms of the modes of waveguide b

$$\left\{ \begin{array}{l} \overline{E}_t(z = 0^+) \\ \overline{H}_t(z = 0^+) \end{array} \right\} = \sum_{j=1}^{N_b} b_j \left\{ \begin{array}{l} \hat{e}_{bj} \\ \hat{h}_{bj} \end{array} \right\} \quad (10.62)$$

where N_a and N_b are the number of modes required in each part to adequately represent the fields. For the junction of figure 10.8 involving a PEC boundary, the boundary conditions at the junction $z = 0$ can be written as

$$\overline{E}_t(z = 0^-) = \begin{cases} 0 & \text{on the metal } (S_a - S_b) \\ \overline{E}_t(z = 0^+) & \text{in the aperture } (S_b) \end{cases} \quad (10.63a)$$

$$\overline{H}_t(z = 0^-) = \overline{H}_t(z = 0^+) \quad \text{in the aperture } (S_b) \quad (10.63b)$$

Substituting the modal expansions for the fields gives

$$\hat{e}_{a1} + \sum_{i=1}^{N_a} a_i \hat{e}_{ai} = \begin{cases} 0 & \text{on the metal} \\ \sum_{j=1}^{N_b} b_j \hat{e}_{bj} & \text{in the aperture } (S_b) \end{cases} \quad (10.64a)$$

$$\hat{h}_{a1} - \sum_{i=1}^{N_a} a_i \hat{h}_{ai} = \sum_{j=1}^{N_b} b_j \hat{h}_{bj} \quad \text{in the aperture} \quad (10.64b)$$

The unknown expansion coefficients can now be determined by exploiting the mode-orthogonality properties. Doting both sides of (10.64a) with \hat{e}_{ak} and integrating over the cross section S_a gives

$$\iint_{S_a} \hat{e}_{a1} \cdot \hat{e}_{ak} dS + \sum_{i=1}^{N_a} a_i \iint_{S_a} \hat{e}_{ai} \cdot \hat{e}_{ak} dS = \sum_{j=1}^{N_b} b_j \iint_{S_b} \hat{e}_{bj} \cdot \hat{e}_{ak} dS \quad (10.65)$$

and subsequently using (10.51) allows us to isolate the k th expansion coefficient a_k as

$$a_k = -\delta_{k1} + \sum_{j=1}^{N_b} b_j \iint_{S_b} \hat{e}_{bj} \cdot \hat{e}_{ak} dS \quad (10.66)$$

Similarly, forming the dot product of (10.64b) with \hat{h}_{bl} and integrating over the aperture S_b allows us to solve for the expansion coefficients b_l as

$$b_l = \sum_{i=1}^{N_a} (\delta_{i1} - a_i) \frac{Z_{bl}}{Z_{ai}} \iint_{S_b} \hat{e}_{bl} \cdot \hat{e}_{ai} dS \quad (10.67)$$

The two equations for a set of $N_a + N_b$ simultaneous equations for the unknown expansion coefficients. For convenience, let

$$P_{jk} = \iint_{S_b} \hat{e}_{bj} \cdot \hat{e}_{ak} dS \quad (10.68)$$

Substituting (10.67) into (10.66) to eliminate the b_l 's gives

$$a_k + \sum_{i=1}^{N_a} a_i \sum_{j=1}^{N_b} \frac{Z_{bj}}{Z_{ai}} P_{jk} P_{ji} = -\delta_{k1} + \sum_{j=1}^{N_b} \frac{Z_{bj}}{Z_{a1}} P_{jk} P_{j1} \quad (10.69)$$

This can be written as a matrix equation

$$\overline{\overline{R}} \cdot \overline{a} = \overline{c} \quad (10.70)$$

where the matrix elements R_{ki} and c_k can be written as

$$R_{ki} = \delta_{ki} + \sum_{j=1}^{N_b} \frac{Z_{bj}}{Z_{ai}} P_{jk} P_{ji}$$

$$c_k = R_{k1} - 2\delta_{k1}$$

Given a certain set of waveguides and junction parameters, the problem then involves computing the matrix $\overline{\overline{R}}$ and inverting it to form the solution vector

$$\overline{a} = \overline{\overline{R}}^{-1} \cdot \overline{c} \quad (10.71)$$

Once the reflection vector \overline{a} is known, the transmission vector \overline{b} can be found using (10.67).

For computational efficiency, it is desirable to do the overlap integral P_{jk} in (10.68) analytically if possible. For problems involving canonical rectangular or cylindrical waveguide structures this is usually possible. There are $N_b \times N_a$ of these integrals, which should only be computed once and stored for later calculation of the R_{ki} . Note also that the ordering of the indices is very important.

Many similar problems have been treated by Wexler [1] including boundary enlargement (as opposed to the boundary reduction problem above) and junctions involving more than two waveguides. The procedures involve, at most, only slight modifications of the above.

10.5.2 Variational Method

In the previous problem, the expansion coefficients describing the aperture fields were found by enforcing the boundary conditions. Equivalent circuits can then be determined using the expansion coefficients for the propagating modes. If the equivalent circuits are expressed directly in terms of the aperture fields, it is often the case that the resulting expression is relatively insensitive to the exact nature of the aperture field. The expression is said to be variational or stationary with respect to small errors in the fields. This is a useful property in practical work, since one can often make an educated guess about the field distribution and consequently deduce a reasonably accurate equivalent circuit for modeling the discontinuity.

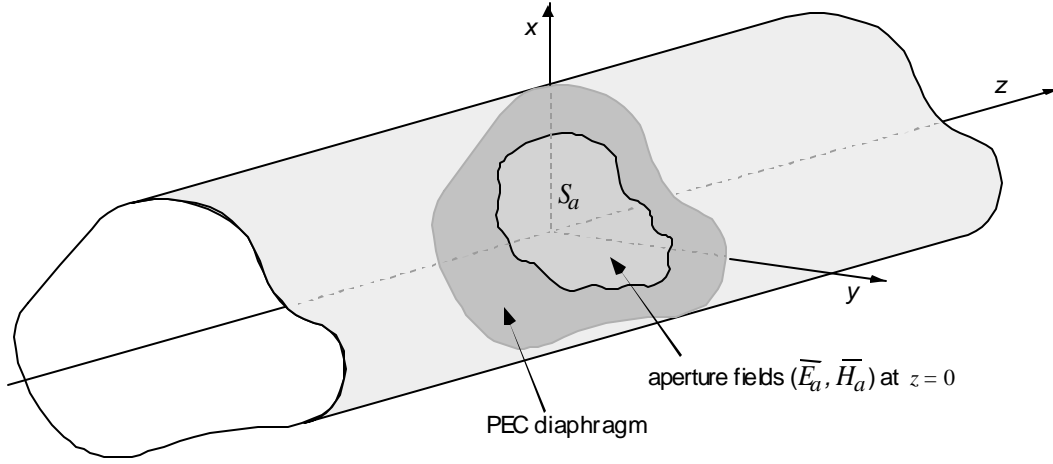


Figure 10.9 Waveguide with a PEC diaphragm at $z = 0$.

Consider a waveguide with a planar metallic diaphragm discontinuity shown in figure 10.9. Here we assume that the waveguide cross-section is the same on both sides for simplicity. If only the dominant mode can propagate in the guide, then we can model this discontinuity by an equivalent shunt admittance

$$Y_d = \frac{-2\Gamma}{1+\Gamma} Y_1 \quad (10.72)$$

where Γ is the dominant-mode reflection coefficient, and Y_1 is dominant mode wave admittance. Using similar ideas from the mode matching method, we expand the fields to the left of the junction at $z = 0^-$ as

$$\bar{E}_t(z = 0^-) = (1 + \Gamma)\hat{e}_1 + \sum_{i=2}^{\infty} a_i \hat{e}_i \quad (10.73a)$$

$$\bar{H}_t(z = 0^-) = (1 - \Gamma)\hat{h}_1 - \sum_{i=2}^{\infty} a_i \hat{h}_i \quad (10.73b)$$

and to the right at $z = 0^+$ as

$$\begin{Bmatrix} \bar{E}_t(z = 0^+) \\ \bar{H}_t(z = 0^+) \end{Bmatrix} = \tau \begin{Bmatrix} \hat{e}_1 \\ \hat{h}_1 \end{Bmatrix} + \sum_{j=2}^{\infty} b_j \begin{Bmatrix} \hat{e}_j \\ \hat{h}_j \end{Bmatrix} \quad (10.74)$$

where τ is the dominant mode transmission coefficient, and (\hat{e}_1, \hat{h}_1) represent the dominant mode. We again enforce continuity of tangential fields at the aperture. Denoting the true aperture field as (\bar{E}_a, \bar{H}_a) , we have

$$\hat{z} \times \bar{E}(z = 0^-) = \begin{cases} 0 & \text{on diaphragm} \\ \hat{z} \times \bar{E}_a & \text{in aperture} \end{cases} = \hat{z} \times \bar{E}(z = 0^+) \quad (10.75a)$$

$$\hat{z} \times \bar{H}(z = 0^-) = \hat{z} \times \bar{H}_a = \hat{z} \times \bar{H}(z = 0^+) \quad \text{in aperture only} \quad (10.75b)$$

Using the mode orthogonality and the boundary condition on tangential \overline{E} field, we can easily find

$$(1 + \Gamma) = \iint_{S_a} (\overline{E}_a \cdot \overline{e}_1) dS \quad a_n = \iint_{S_a} (\overline{E}_a \cdot \overline{e}_n) dS \quad n > 1 \quad (10.76)$$

and

$$\tau = \iint_{S_a} (\overline{E}_a \cdot \overline{e}_1) dS \quad b_n = \iint_{S_a} (\overline{E}_a \cdot \overline{e}_n) dS \quad n > 1 \quad (10.77)$$

where the integration is over the aperture, S_a , not the entire waveguide cross section. Note that $1 + \Gamma = \tau$, as expected from the equivalent circuit, and that $a_n = b_n$ for $n > 1$. Using the boundary condition on tangential \overline{H} fields we can write

$$(1 - \Gamma)\overline{h}_1 - \sum_{n=2}^{\infty} a_n \overline{h}_n = (1 + \Gamma)\overline{h}_1 + \sum_{n=2}^{\infty} b_n \overline{h}_n \quad (10.78)$$

Substituting for a_n and b_n and using (10.41) gives

$$-2\Gamma \frac{\overline{e}_1}{Z_1} = 2 \sum_{n=2}^{\infty} \frac{\overline{e}_n}{Z_n} \iint_{S_a} (\overline{E}_a \cdot \overline{e}_n) dS \quad (10.79)$$

Forming the dot product of both sides with \overline{E}_a and integrating over the waveguide cross section gives

$$-\frac{2\Gamma}{Z_1} \iint_{S_a} (\overline{E}_a \cdot \overline{e}_1) dS = 2 \sum_{n=2}^{\infty} \frac{1}{Z_n} \left[\iint_{S_a} (\overline{E}_a \cdot \overline{e}_n) dS \right]^2$$

This can be solved for Γ . Using this result and (10.76) we can then express (10.72) as

$$\frac{Y_d}{Y_1} = \frac{-2\Gamma}{1 + \Gamma} = \frac{2 \sum_{n=2}^{\infty} \frac{1}{Z_n} \left[\iint_{S_a} (\overline{E}_a \cdot \overline{e}_n) dS \right]^2}{\frac{1}{Z_1} \left[\iint_{S_a} (\overline{E}_a \cdot \overline{e}_1) dS \right]^2} \quad (10.80)$$

This is the variational expression for Y_d . We wish to show that (10.80) is stationary with respect to a first order variation in \overline{E}_a . If the aperture field is perturbed by a small correction $\overline{E}_a \rightarrow \overline{E}_a + \Delta\overline{E}_a$, the admittance will change to $Y_d + \Delta Y_d$. Making these substitutions in (10.80), cross multiplying, and keeping only first-order terms leaves

$$\begin{aligned} \frac{\Delta Y_d}{Y_1} \frac{1}{Z_1} \left[\iint_{S_a} (\overline{E}_a \cdot \overline{e}_1) dS \right]^2 &+ \frac{Y_d}{Y_1} \frac{2}{Z_1} \iint_{S_a} (\overline{E}_a \cdot \overline{e}_1) dS \iint_{S_a} (\Delta\overline{E}_a \cdot \overline{e}_1) dS \\ &= \sum_{n=2}^{\infty} \frac{4}{Z_n} \iint_{S_a} (\overline{E}_a \cdot \overline{e}_n) dS \iint_{S_a} (\Delta\overline{E}_a \cdot \overline{e}_n) dS \end{aligned} \quad (10.81)$$

Taking the dot product of both sides of (10.79) with $\Delta\overline{E}_a$ and integrating gives

$$\frac{2}{Z_1} \iint_{S_a} (\Delta\overline{E}_a \cdot \overline{e}_1) dS = -\frac{2}{\Gamma} \sum_{n=2}^{\infty} \frac{1}{Z_n} \iint_{S_a} (\overline{E}_a \cdot \overline{e}_n) dS \iint_{S_a} (\Delta\overline{E}_a \cdot \overline{e}_n) dS$$

and since (10.76) allows us to write

$$\frac{Y_d}{Y_1} = \frac{-2\Gamma}{1+\Gamma} = \frac{-2\Gamma}{\iint_{S_a} (\bar{E}_a \cdot \bar{e}_1) dS}$$

then the last two terms in (10.81) are identical, and

$$\Delta Y_d \left[\iint_{S_a} (\bar{E}_a \cdot \bar{e}_1) dS \right]^2 = 0 \quad (10.82)$$

and therefore $\Delta Y_d = 0$. The first-order correction to Y_d is zero for a small error in \bar{E}_a . In other words, an approximate “guess” at \bar{E}_a will yield a more accurate solution for Y_d .

10.6 INTEGRAL EQUATION FORMALISM FOR WAVEGUIDE SCATTERING

The problems treated by model expansion in previous sections dealt exclusively with planar obstacles or discontinuities, and consequently only transverse fields were required. A more general scattering problem is shown in figure 10.10, where an arbitrarily shaped conducting obstacle placed in the waveguide. Incident fields on the obstacle will excite currents which will in turn reradiate fields into the waveguide. We can treat this problem rigorously using the dyadic Green’s function developed in the previous section. The approach is typical of most scattering problems.

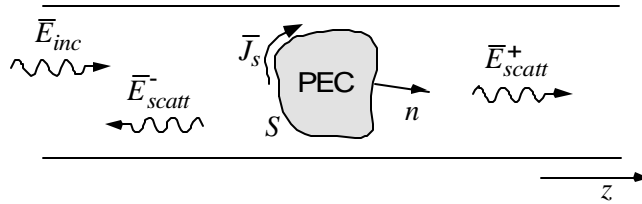


Figure 10.10 Scattering from a PEC obstacle in a waveguide.

Let the incident field be \bar{E}_{inc} , and the scattered field be \bar{E}_{scatt} . On the surface of the obstacle the *total* tangential fields must vanish, so we write

$$\hat{n} \times (\bar{E}_{inc} + \bar{E}_{scatt}) = 0 \quad \text{on } S \quad (10.83)$$

The scattered fields are computed from the surface currents using the Green’s function,

$$\bar{E}_{scatt} = -j\omega\mu \iint_S \bar{\bar{G}}(\bar{r}, \bar{r}') \cdot \bar{J}(\bar{r}') dS' \quad (10.84)$$

One possible representation for G has been found already

$$\bar{\bar{G}}(\bar{r}, \bar{r}') = -\frac{1}{2j\omega\mu} \begin{cases} \sum_n Z_n \bar{E}_n^+(\bar{r}) \bar{E}_n^-(\bar{r}') & z > z' \\ \sum_n Z_n \bar{E}_n^-(\bar{r}) \bar{E}_n^+(\bar{r}') & z < z' \end{cases} \quad (10.85)$$

So we have

$$\hat{n} \times \bar{E}_{inc}(\bar{r}) + \oint\!\!\!\oint_s [\hat{n} \times \bar{G}(\bar{r}, \bar{r}')] \cdot \bar{J}(\bar{r}') dS' = 0 \quad \underline{\bar{r} \text{ on } S} \quad (10.86)$$

Assuming the incident field is specified, this is an integral equation for the unknown current. We can solve this using the Method-of-Moments technique, expanding the surface current in terms of a basis set $\bar{J}_n(\bar{r}')$,

$$\bar{J}(\bar{r}') = \sum_{n=1}^N I_n \bar{J}_n(\bar{r}') \quad (10.87)$$

Substituting this gives

$$\sum_{n=1}^N I_n \oint\!\!\!\oint_s [\hat{n} \times \bar{G}(\bar{r}, \bar{r}')] \cdot \bar{J}_n(\bar{r}') dS' = -\hat{n} \times \bar{E}_{inc}(\bar{r}) \quad \underline{\bar{r} \text{ on } S} \quad (10.88)$$

Enforcing this equation at N different points on S (the point-matching method) gives a matrix equation for the unknown current coefficients. Alternatively we could use a Galerkin method where (10.88) is dotted with \bar{J}_m in place of the $\hat{n} \times$ (this is permissible since \bar{J}_m is tangential to the surface) and integrated over the surface S to give

$$\sum_{n=1}^N I_n \oint\!\!\!\oint_s \oint\!\!\!\oint_s \bar{J}_m(\bar{r}) \cdot \bar{G}(\bar{r}, \bar{r}') \cdot \bar{J}_n(\bar{r}') dS' dS = - \oint\!\!\!\oint_s \bar{E}_{inc}(\bar{r}) \cdot \bar{J}_m(\bar{r}) dS \quad (10.89)$$

where $m = 1, 2, \dots, N$. This gives better accuracy (usually) for a given basis, but the double surface integrals are quite time consuming when done numerically.

In any case, once the I_n are known, then we can compute the coupling to any other mode in the waveguide, and hence determine the circuit parameters. For example, to find S_{11} for the dominant mode, we take the incident excitation to be from the left

$$\bar{E}_{inc} = \bar{E}_1^+ \quad (10.90)$$

Then find I_n using the procedure above. The scattered field back toward the source ($z < z'$) is

$$\bar{E}_{scatt}^- = \sum_n a_n \bar{E}_n^-(\bar{r}) = \frac{1}{2} \sum_n \bar{E}_n^-(\bar{r}) \oint\!\!\!\oint_s \bar{E}_n^+(\bar{r}') \cdot \bar{J}(\bar{r}') dS' \quad (10.91)$$

and therefore the reflection coefficient is

$$S_{11} = \frac{1}{2} \sum_{n=1}^N I_n \oint\!\!\!\oint_s \bar{E}_1^+(\bar{r}') \cdot \bar{J}_n(\bar{r}') dS \quad (10.92)$$

Other S -parameters are found similarly.

REFERENCES

- [1] A. Wexler, "Solution of waveguide discontinuities by modal analysis", *IEEE Trans. Microwave Theory Tech.*, vol. MTT-15, pp. 508-517, Sept 1967.
- [2] W.W. Hansen, "A new type of expansion in radiation problems", *Phys. Rev.*, vol. 47, Jan 15, 1935, pp. 139-143.