10.3 MODAL EXPANSION OF WAVEGUIDE FIELDS

It will prove useful to adopt a standardized notation for keeping track of field variables. We adopt the following: the *true* fields (satisfying Maxwell's equations) are labeled with uppercase Roman letters, $(\overline{E}, \overline{H})$. A "$\pm$" superscript on a field variable denotes waves propagating in the $\pm\hat{z}$ direction according to $e^{\mp\gamma z}$. We always assume propagation parallel to the $\hat{z}$-axis; coordinates in the transverse plane are given by the transverse position vector, $\overline{r}_t$. The spatial variation of the fields in the transverse plane is described by the functions $(\overline{e}, \overline{h})$, which only depend on $\overline{r}_t$. The lowercase notation reminds us that $\overline{e}$ and $\overline{h}$ alone do not satisfy Maxwell's equations.

Therefore, the nth mode of a waveguide can be described in the following equivalent ways,

$$\begin{aligned}
\overline{E}_n^{\pm}(\overline{r}_t, z) &= \overline{E}_{tn}^{\pm}(\overline{r}_t, z) + \hat{z}E_{zn}^{\pm}(\overline{r}_t, z) & \overline{H}_n^{\pm}(\overline{r}_t, z) &= \overline{H}_{tn}^{\pm}(\overline{r}_t, z) + \hat{z}H_{zn}^{\pm}(\overline{r}_t, z) \\
&= \overline{e}_n^{\pm}(\overline{r}_t)e^{\mp\gamma z} & &= \overline{h}_n^{\pm}(\overline{r}_t)e^{\mp\gamma z} \\
&= \left[\overline{e}_n(\overline{r}_t) \pm \hat{z}e_{zn}(\overline{r}_t)\right] e^{\mp\gamma z} & &= \left[\pm\overline{h}_n(\overline{r}_t) + \hat{z}h_{zn}(\overline{r}_t)\right] e^{\mp\gamma z}
\end{aligned} \tag{10.40}$$

Note the difference between the transverse field vectors $(\overline{e}_n^{\pm}, \overline{h}_n^{\pm})$ and $(\overline{e}_n, \overline{h}_n)$; the former represents both transverse and longitudinal components of the field, whereas the latter represents only the transverse components. Furthermore, the transverse field functions $(\overline{e}_n, \overline{h}_n)$ and longitudinal field functions $(\overline{e}_{zn}, \overline{h}_{zn})$ are defined as positive for waves propagating in the positive $\hat{z}$ direction; for waves travelling in the $-\hat{z}$ direction, the signs are explicitly written as shown in the last line of (10.40). The transverse field functions are thus defined such that

$$\overline{h}_n = \frac{1}{Z_n}\hat{z} \times \overline{e}_n \qquad \overline{e}_n = Z_n\overline{h}_n \times \hat{z} \tag{10.41}$$

The mode index n can represent any possible waveguide mode, TE, TM, or TEM. For computation, the modes are usually arranged so that the sequence $n = 0, 1, 2, 3 \ldots$ corresponds to modes of increasing cutoff frequency, with $n = 0$ always corresponding to the TEM mode (if it exists). An arbitrary field distribution in the waveguide can then be written as

$$\left\{ \begin{array}{c} \overline{E}(\overline{r}) \\ \overline{H}(\overline{r}) \end{array} \right\} = \sum_n \left[a_n \left\{ \begin{array}{c} \overline{E}_n^{+} \\ \overline{H}_n^{+} \end{array} \right\} + b_n \left\{ \begin{array}{c} \overline{E}_n^{-} \\ \overline{H}_n^{-} \end{array} \right\} \right] \tag{10.42}$$

This assumes the modes form a complete set. We will not attempt to prove this, but it is generally the case.

10.3.1 Orthogonality

Earlier we showed that the eigenmodes of (10.8) are orthogonal in the sense of (**??**). We now demonstrate a corresponding orthogonality property in terms of the transverse field functions directly, in a way that will extend to the case of waveguides with imperfect conducting enclosures.

Consider two linearly independent modes of a waveguide, $(\overline{E}_m, \overline{H}_m)$ and $(\overline{E}_n, \overline{H}_n)$. For each mode there is a corresponding propagation constant, γ_m and γ_n. Assuming that the waveguide is bidirectional (reciprocal), we can write the modes traveling in the $\pm\hat{z}$ direction as

$$\begin{aligned} \overline{E}_m^{\pm} &= (\overline{e}_m \pm \hat{z} e_{zm})\, e^{\mp\gamma_m z} & \overline{E}_n^{\pm} &= (\overline{e}_n \pm \hat{z} e_{zn})\, e^{\mp\gamma_n z} \\ \overline{H}_m^{\pm} &= \left(\pm\overline{h}_m + \hat{z} h_{zm}\right) e^{\mp\gamma_m z} & \overline{H}_n^{\pm} &= \left(\pm\overline{h}_n + \hat{z} h_{zn}\right) e^{\mp\gamma_n z} \end{aligned} \tag{10.43}$$

Each mode satisfies Maxwell's equations (source free) separately,

$$\begin{aligned} \nabla \times \overline{E}_m^{\pm} &= -\jmath\omega\mu \overline{H}_m^{\pm} & \nabla \times \overline{H}_m^{\pm} &= \jmath\omega\epsilon \overline{E}_m^{\pm} \\ \nabla \times \overline{E}_n^{\pm} &= -\jmath\omega\mu \overline{H}_n^{\pm} & \nabla \times \overline{H}_n^{\pm} &= \jmath\omega\epsilon \overline{E}_n^{\pm} \end{aligned} \tag{10.44}$$

These can be manipulated to give

$$\begin{aligned} \overline{H}_m^{\pm} \cdot \left(\nabla \times \overline{E}_n^{\pm}\right) - \overline{H}_n^{\pm} \cdot \left(\nabla \times \overline{E}_m^{\pm}\right) &= -\jmath\omega\mu \left(\overline{H}_m^{\pm} \cdot \overline{H}_n^{\pm} - \overline{H}_n^{\pm} \cdot \overline{H}_m^{\pm}\right) = 0 \\ \overline{E}_m^{\pm} \cdot \left(\nabla \times \overline{H}_n^{\pm}\right) - \overline{E}_n^{\pm} \cdot \left(\nabla \times \overline{H}_m^{\pm}\right) &= \jmath\omega\epsilon \left(\overline{E}_m^{\pm} \cdot \overline{E}_n^{\pm} - \overline{E}_n^{\pm} \cdot \overline{E}_m^{\pm}\right) = 0 \end{aligned}$$

Substracting and using (A.47) gives

$$\nabla \cdot \left[\overline{E}_m^{\pm} \times \overline{H}_n^{\pm} - \overline{E}_n^{\pm} \times \overline{H}_m^{\pm}\right] = 0 \tag{10.45}$$

Since $\nabla = \nabla_t + \hat{z}\frac{\partial}{\partial z}$ we have

$$\nabla_T \cdot \left[\overline{E}_m^{\pm} \times \overline{H}_n^{\pm} - \overline{E}_n^{\pm} \times \overline{H}_m^{\pm}\right] + \frac{\partial}{\partial z}\left[\hat{z} \cdot \left(\overline{E}_m^{\pm} \times \overline{H}_n^{\pm}\right) - \hat{z}\left(\overline{E}_n^{\pm} \times \overline{H}_m^{\pm}\right)\right] = 0 \tag{10.46}$$

Integrating both sides of (10.46) over the waveguide cross section S, and making use of the two-dimensional version of the divergence theorem, (A.98),

$$\iint_S \nabla_T \cdot \overline{A}\, dS = \oint_C (\hat{n} \cdot \overline{A}) d\ell \tag{A.98}$$

we find that

$$\iint_S \nabla_t \cdot \left[\overline{E}_m^{\pm} \times \overline{H}_n^{\pm} - \overline{E}_n^{\pm} \times \overline{H}_m^{\pm}\right] dS = \oint_C \hat{n} \cdot \left[\overline{E}_m^{\pm} \times \overline{H}_n^{\pm} - \overline{E}_n^{\pm} \times \overline{H}_m^{\pm}\right] d\overline{\ell} \tag{10.47}$$

where C is a path around the waveguide enclosure in a cross-sectional plane, and $\hat{n}$ is the unit vector normal to C. If the waveguide is constructed of PEC or PMC material, then the path integral evaluates to zero since from (A.38), $\hat{n} \cdot (\overline{E} \times \overline{H}) = \overline{H} \cdot (\hat{n} \times \overline{E}) = \overline{E} \cdot (\overline{H} \times \hat{n})$. Then (10.46) and (10.47) lead to

$$\iint_S \frac{\partial}{\partial z}\left[\hat{z} \cdot \left(\overline{E}_m^{\pm} \times \overline{H}_n^{\pm}\right) - \hat{z}\left(\overline{E}_n^{\pm} \times \overline{H}_m^{\pm}\right)\right] = 0 \tag{10.48}$$

If we take both modes propogating in the $+\hat{z}$ direction (top sign) then

$$\frac{\partial}{\partial z}\left[\hat{z} \cdot \left(\overline{E}_m^{+} \times \overline{H}_n^{+}\right)\right] = -\left(\gamma_m + \gamma_n\right)\left(\overline{e}_{tm} \times \overline{h}_{tn}\right)$$

Alternatively, if the m_{th} mode travels in the $-\hat{z}$ direction then

$$\frac{\partial}{\partial z}\left[\hat{z}\cdot\left(\overline{E}_m^-\times\overline{H}_n^+\right)\right] = -\left(\gamma_m-\gamma_n\right)\left(\overline{e}_{tm}\times\overline{h}_{tn}\right).$$

We can therefore generate the two following equations from (10.48)

$$(\gamma_m+\gamma_n)\quad\iint_S\left[\overline{e}_{tm}\times\overline{h}_{tn}-\overline{e}_{tn}\times\overline{h}_{tm}\right]\cdot d\overline{S}=0 \qquad (10.49a)$$

$$(-\gamma_m+\gamma_n)\quad\iint_S\left[\overline{e}_{tm}\times\overline{h}_{tn}+\overline{e}_{tn}\times\overline{h}_{tm}\right]\cdot d\overline{S}=0 \qquad (10.49b)$$

For non-degenerate modes with $m\neq n$ and $\gamma_n\neq\gamma_m$, we can solve (10.49) simultaneously giving

$$\iint_S\left(\overline{e}_{tm}\times\overline{h}_{tn}\right)\cdot d\overline{S}=0 \qquad (10.50)$$

This expresses the orthogonality property of transverse fields in different modes. For two different modes that share a common eigenvalue this proof is inconclusive, although it is often the case that such degenerate modes are still orthogonal in the sense of (10.50). In any case, degenerate modes can always be linearly superposed to form orthogonal modes using the Gram-Schmidt procedure, as described in Appendix B.

10.3.2 Normalization

There is more than one possible normalization condition. It will be assumed that the transverse electric fields are orthonormal in the sense of

$$\iint_S\hat{e}_m\cdot\hat{e}_n\,dS=\delta_{mn} \qquad (10.51)$$

where S is the waveguide cross section, and the circumflex ($\hat{}$) denotes a normalized vector. We can rewrite (10.51) by substituting (10.41), giving

$$\iint_S\hat{h}_m\cdot\hat{h}_n\,dS=\frac{1}{Z_n^2}\delta_{mn} \qquad (10.52)$$

or also

$$\iint_S(\hat{e}_m\times\hat{h}_n)\cdot\hat{z}\,dS=\frac{1}{Z_n}\delta_{mn} \qquad (10.53)$$

These results all depend on the choice of normalization (10.51); problem xx treats another possible normalization condition. Using the orthogonality properties of the scalar wavefunctions, it can be shown that the above normalization conditions also lead to the following results involving the longitudinal field functions

$$\text{TE waves:}\qquad\iint_S\hat{h}_{zm}\cdot\hat{h}_{zn}\,dS=\frac{k_c^2}{Z_n^2\gamma_n^2}\delta_{mn}=\left(\frac{k_c}{\jmath\omega\mu}\right)^2\delta_{mn} \qquad (10.54a)$$

$$\text{TE waves:}\qquad\iint_S\hat{e}_{zm}\cdot\hat{e}_{zn}\,dS=\frac{k_c^2}{\gamma_n^2}\delta_{mn} \qquad (10.54b)$$

Note that the normalization process may change the units of the normalized field variables and associated expansion coefficients. If (10.51) is used, then $\hat{e}_n$ will have the units of inverse meters [1/m], and a_n and b_n will have the units of Volts [V].