AUXILIARY POTENTIALS AND TE/TM FIELD DECOMPOSITION

The time-harmonic Maxwell's equations are

$$\nabla \cdot \overline{D} = \rho_e \tag{1a}$$
$$\nabla \cdot \overline{B} = \rho_m \tag{1b}$$
$$\nabla \times \overline{E} = -\overline{M} - \jmath\omega\overline{B} \tag{1c}$$
$$\nabla \times \overline{H} = \overline{J} + \jmath\omega\overline{D} \tag{1d}$$

which are consistent with charge conservation, expressed through the continuity equations

$$\nabla \cdot \overline{J} + \jmath\omega\rho_e = 0 \tag{2a}$$
$$\nabla \cdot \overline{M} + \jmath\omega\rho_m = 0 \tag{2b}$$

The two curl equations in (1) can be decoupled by taking the curl of both sides and making some obvious substitutions, resulting in two inhomogeneous vector wave equations, which (for simple media) are

$$\nabla \times \nabla \times \overline{E} - k^2\overline{E} = -\jmath\omega\mu\overline{J} - \nabla \times \overline{M} \tag{3a}$$
$$\nabla \times \nabla \times \overline{H} - k^2\overline{H} = \nabla \times \overline{J} - \jmath\omega\epsilon\overline{M} \tag{3b}$$

where $k^2 = \omega^2\mu\epsilon$. Using the identity $\nabla \times \nabla \times \overline{A} = \nabla\nabla \cdot \overline{A} - \nabla^2\overline{A}$, substituting the remaining two divergence equations from (1), and rewriting the charge densities in terms of the currents through (2), we get

$$\nabla^2\overline{E} + k^2\overline{E} = \jmath\omega\mu\left[\overline{J} + \frac{1}{k^2}\nabla(\nabla \cdot \overline{J})\right] + \nabla \times \overline{M} \tag{4a}$$

$$\nabla^2\overline{H} + k^2\overline{H} = -\nabla \times \overline{J} + \jmath\omega\epsilon\left[\overline{M} + \frac{1}{k^2}\nabla(\nabla \cdot \overline{M})\right] \tag{4b}$$

These equations are just a restatement of Maxwell's equations in simple media (note that all four Maxwell's equations were used in the derivation), but in a potentially more convenient form for analysis. Both are in the same form, referred to as the inhomogeneous vector Helmholtz equation. The right hand side of each—the "forcing function" in the jargon of differential equations—involves only the current densities, which are assumed known. The equations are decoupled, each involving only one field variable. In rectangular coordinates, the left hand side of each equation reduces to three scalar equations of exactly the same form.

In later work we will discuss the general solution of equations like (4) using the formalism of dyadic Green's functions. Before doing that, we will examine alternative expressions of Maxwell's equations through the introduction of auxiliary potential functions in place of the electric and magnetic field quantities. This approach is not strictly necessary, but is often useful in simplifying the analysis and/or notation in certain types of problems.

1 VECTOR POTENTIALS

Consider electric and magnetic sources separately. When only electric sources are present ($\rho_m = \overline{M} = 0$) then $\nabla \cdot \overline{B} = 0$, and since the divergence of any curl is identically zero (A.51) we can express $\overline{B}$ as the curl of another vector function

$$B = \nabla \times \overline{A} \tag{5}$$

Substituting (5) into (1c) gives

$$\nabla \times \overline{E} = -\jmath\omega\nabla \times \overline{A} \quad \Rightarrow \quad \nabla \times (\overline{E} + \jmath\omega\overline{A}) = 0$$

Since the curl of any gradient is identically zero (A.50) then we can write

$$\overline{E} + \jmath\omega\overline{A} = -\nabla\phi_e \tag{6}$$

where ϕ_e is an arbitrary scalar function. The negative sign in (6) is chosen so that the scalar function ϕ_e reduces to the electrostatic potential in the limit of $\omega \to 0$. Note that $\overline{A}$ and ϕ_e are not uniquely determined by (5) and (6), since we can define a new set of potentials $(\overline{A}', \phi_e')$ which will produce the same fields via the *gauge transformation*

$$\overline{A}' \to \overline{A} + \nabla\xi \qquad \phi_e' \to \phi_e - \jmath\omega\xi$$

The scalar function ξ is arbitrary, so there are an infinite number of potentials $(\overline{A}, \phi_e)$ which will produce a given field distribution; this flexibility will be exploited to simplify the analysis. Specification of ξ is referred to as the choice of "gauge", and therefore $\overline{E}$ and $\overline{B}$ are said to be *gauge invariant*. This is an indirect consequence of the *Helmholtz theorem*, which states that a vector function is uniquely determined only by specifying *both* it's divergence and curl (see Appendix A). So far we have only specified the curl of $\overline{A}$ via (5). We are free to choose the $\nabla \cdot \overline{A}$ arbitrarily, which amounts to fixing the value of ξ.

1.1 Lorentz Gauge

For simple dielectrics, substitution of (5) and (6) into (1a) and (1d) gives

$$\nabla \times \nabla \times \overline{A} = \mu\overline{J} + \jmath\omega\mu\epsilon(-\jmath\omega\overline{A} - \nabla\phi_e)$$
$$\nabla \cdot (-\jmath\omega\overline{A} - \nabla\phi_e) = \rho_e/\epsilon$$

Using the identity $\nabla \times \nabla \times \overline{A} = \nabla\nabla \cdot \overline{A} - \nabla^2\overline{A}$ and rearranging gives

$$\nabla^2\overline{A} + k^2\overline{A} - \nabla(\nabla \cdot \overline{A} + \jmath\omega\mu\epsilon\phi_e) = -\mu\overline{J} \tag{7a}$$
$$\nabla^2\phi_e + \jmath\omega\nabla \cdot \overline{A} = -\rho_e/\epsilon \tag{7b}$$

This form suggests the following choice for the divergence of $\overline{A}$, called the *Lorentz gauge*,

$$\nabla \cdot \overline{A} \equiv -\jmath\omega\mu\epsilon\phi_e \tag{8}$$

which converts (7) into a pair of inhomogeneous Helmholtz equations

$$\nabla^2\overline{A} + k^2\overline{A} = -\mu\overline{J} \tag{9a}$$
$$\nabla^2\phi_e + k^2\phi_e = -\rho_e/\epsilon \tag{9b}$$

Equations (9a) and (9b) are a restatement of Maxwell's equations in terms of a single vector function $\overline{A}$ and the scalar function ϕ_e, and represent four scalar equations of *exactly the same form.* In fact, it is only necessary to solve (9a), since both $\overline{E}$ and $\overline{H}$ can be expressed in terms of $\overline{A}$ using (8) as

$$\overline{H} = \frac{1}{\mu}\nabla \times \overline{A} \tag{10a}$$

$$\overline{E} = -\jmath\omega\left[\overline{A} + \frac{1}{k^2}\nabla\nabla\cdot\overline{A}\right] \tag{10b}$$

Usually when the current distribution is known it is much easier to solve for the vector potential $\overline{A}$, and subsequently for the fields through (10), than to attempt a solution of Maxwell's equations directly. Later we will derive far-field expressions which also make advantageous use of the vector potential formulation. However, in the solution of boundary value problems, the vector potentials are of little assistance since the boundary conditions are specified in terms of $\overline{E}$ and $\overline{H}$. This will be discussed later.

Another set of potentials can be derived for the fields produced by magnetic sources. In the absence of electric sources ($\rho_e = \overline{J} = 0$), we can write $\overline{D} = \nabla \times \overline{F}$, where $\overline{F}$ is an electric vector potential. Following exactly the same steps that lead to (9a), we find a $\overline{F}$ is governed by a similar inhomogeneous Helmholtz equation

$$\nabla^2\overline{F} + k^2\overline{F} = -\epsilon\overline{M} \tag{11}$$

The fields produced by these magnetic currents are then calculated from $\overline{F}$ as follows

$$\overline{E} = -\frac{1}{\epsilon}\nabla \times \overline{F} \tag{12a}$$

$$\overline{H} = -\jmath\omega\left[\overline{F} + \frac{1}{k^2}\nabla\nabla\cdot\overline{F}\right] \tag{12b}$$

In the general case where both electric and magnetic sources are present, the principle of superposition allows us to write the solution as a the sum of fields produced by electric and magnetic sources separately, so that in general,

$$\overline{E} = -\jmath\omega\left[\overline{A} + \frac{1}{k^2}\nabla\nabla\cdot\overline{A}\right] - \frac{1}{\epsilon}\nabla \times \overline{F} \tag{13a}$$

$$\overline{H} = \frac{1}{\mu}\nabla \times \overline{A} - \jmath\omega\left[\overline{F} + \frac{1}{k^2}\nabla\nabla\cdot\overline{F}\right] \tag{13b}$$

1.2 Coulomb Gauge

The Lorentz condition (8) is not the only choice of gauge that is useful, but it is the one most commonly encountered in engineering electromagnetics. Another choice that is widely used in quantum electrodynamics is the *Coulomb gauge*, which is characterized by the choice

$$\nabla\cdot\overline{A} = 0 \tag{14}$$

From (7) the potentials must then satisfy

$$\nabla^2\overline{A} + k^2\overline{A} = -\mu\overline{J} + \jmath\omega\mu\epsilon\nabla\phi_e \tag{15a}$$

$$\nabla^2\phi_e = -\rho_e/\epsilon \tag{15b}$$

The scalar potential satisfies Poisson's equation, which has the well-known solution

$$\phi_e(\overline{r}) = \iiint \frac{\rho_e(\overline{r}')}{4\pi\epsilon|\overline{r} - \overline{r}'|} dV' \tag{16}$$

and which explains the origin of the name "Coulomb" gauge.

The equation for the vector potential can again be put in the form of a vector Helmoholtz equation with the help of the Helmholtz theorem (Appendix A), which states that any vector function (the current density $\overline{J}$ in this case) can be represented as the sum of a rotational and irrotational part,

$$\overline{J}(\overline{r}) = \underbrace{-\nabla \iiint \frac{\nabla' \cdot \overline{J}(\overline{r}')}{4\pi|\overline{r} - \overline{r}'|} dV'}_{\overline{J}_\ell} + \underbrace{\nabla \times \iiint \frac{\nabla' \times \overline{J}(\overline{r}')}{4\pi|\overline{r} - \overline{r}'|} dV'}_{\overline{J}_t} \tag{17}$$

In this context, $\overline{J}_\ell$ and $\overline{J}_t$ are often called the *longitudinal* and *transverse* parts of the currents distribution, which refer to the parts of the distribution that are either parallel or perpendicular to the direction from the source element to an observation point. Using the continuity equation $\nabla \cdot \overline{J} + \jmath\omega\rho_e = 0$, we find that

$$\overline{J}_\ell = \jmath\omega\epsilon\nabla\phi_e \tag{18}$$

and therefore the vector potential $\overline{A}$ must satisfy

$$\nabla^2\overline{A} + k^2\overline{A} = -\mu\overline{J}_t \tag{19}$$

As we will see later, it is only the transverse part of the current distribution $\overline{J}_t$ which will contribute to far-field radiation, so this gauge conveniently separates the radiation field from the nonradiative contribution found from ϕ_e. For this reason the Coulomb gauge is also frequently referred to as the *radiation gauge* or *transverse gauge*. From a purely mathematical point of view, the equations for the vector potential have the same form in both the Coulomb gauge and the Lorentz gauge, so neither gauge is preferred in the sense of simplifying the mathematics. Naturally, both gauges should give exactly the same fields for a given current distribution.

1.3 Hertz vectors in the Lorentz Gauge

We noted that (9) is a restatement of Maxwell's equations in terms of the potentials $\overline{A}$ and ϕ_e (assuming no magnetic sources), which involve four scalar functions. Actually, these equations can be represented in terms of a single vector function, since one of the four scalar functions is determined by the gauge condition. For example, if we define $\overline{A}$ and ϕ_e in terms of the vector function $\overline{\Pi}_e$ as

$$\overline{A} \equiv \jmath\omega\mu\epsilon\overline{\Pi}_e \qquad \phi_e = -\nabla \cdot \overline{\Pi}_e \tag{20}$$

then the Lorentz gauge is satisfied and the equations (9) governing the potentials both reduce to

$$\nabla^2\overline{\Pi}_e + k^2\overline{\Pi}_e = -\frac{\overline{J}}{\jmath\omega\epsilon} \tag{21}$$

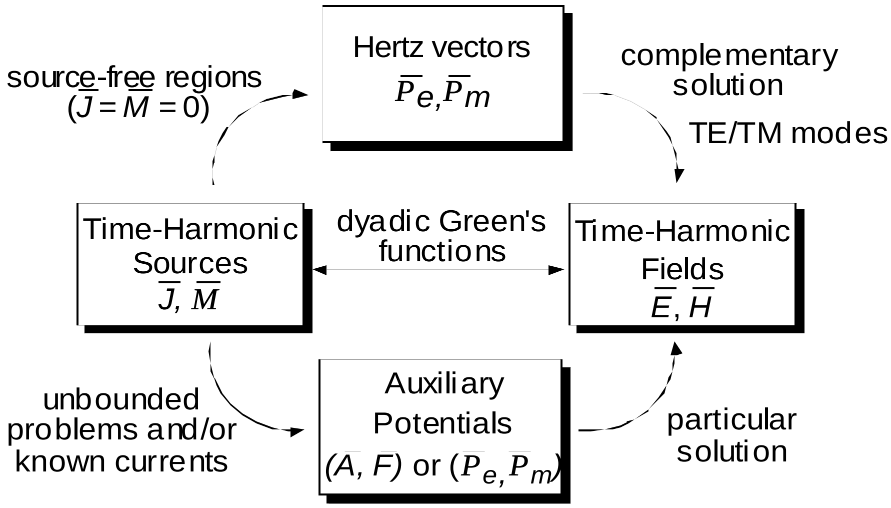

Figure 1 Use of auxiliary potentials for problem solving.

The vector $\overline{\Pi}_e$ is called the *electric Hertz vector*. We can similarly write the vector potential $\overline{F}$ in terms of a *magnetic Hertz vector*, $\overline{F} = \jmath\omega\mu\epsilon\overline{\Pi}_m$, leading to a correspondingly similar governing equation

$$\nabla^2\overline{\Pi}_m + k^2\overline{\Pi}_m = -\frac{\overline{M}}{\jmath\omega\mu} \tag{22}$$

The fields are then given by

$$\overline{E}(\overline{r}) = \nabla\nabla\cdot\overline{\Pi}_e + k^2\overline{\Pi}_e - \jmath\omega\mu\nabla\times\overline{\Pi}_m \tag{23a}$$

$$\overline{H}(\overline{r}) = \jmath\omega\epsilon\nabla\times\overline{\Pi}_e + \nabla\nabla\cdot\overline{\Pi}_m + k^2\overline{\Pi}_m \tag{23b}$$

As the reader can observe, for time-harmonic fields there is essentially no difference between the Hertz vectors $(\overline{\Pi}_e, \overline{\Pi}_m)$ and the set $(\overline{A}, \overline{F})$. Both sets of potentials satisfy equations of exactly the same form, and the two sets of potentials are related by a simple scalar proportionality factor. However, in the literature the Hertz vector notation is often reserved for fields in *source-free* regions, whereas the set $(\overline{A}, \overline{F})$ is typically associated with a particular solution of Maxwell's equations directly in terms of the currents. This conventional use of the various auxiliary potentials is summarized in figure 1, which shows various paths to a field solution given a certain current distribution. Later we will use the Hertz vector in a discussion of TE/TM field decomposition in source-free regions, which will be instrumental in our development of waveguide theory and a vector field expansion in spherical coordinates.

1.4 Green's Functions for the Scalar Helmoholtz Equation

Clearly it is important to find a solution to the inhomogeneous Helmholtz equation. Consider the vector potentials governed by (9a) and (11). In rectangular coordinates, all cartesian components of the potentials also satisfy the same inhomogeneous scalar wave equation; for example, A_x is governed by

$$\nabla^2 A_x(\overline{r}) + k^2 A_x(\overline{r}) = -\mu J_x(\overline{r})$$

This can be written concisely in operator notation as

$$\mathcal{L}\, A_x = -\mu J_x(\overline{r}) \tag{24}$$

where $\mathcal{L}$ is the linear vector differential operator $\mathcal{L} = \nabla^2 + k^2$, whcih can be called the Helmholtz operator. Suppose we have a solution to this equation for the special case of a point source located at $\overline{r}'$; that is, we have a scalar function $g(\overline{r},\overline{r}')$ which is a known solution to

$$\mathcal{L}\, g(\overline{r},\overline{r}') = -\delta(\overline{r} - \overline{r}') \tag{25}$$

(the negative sign in front of the delta function is used by convention). Note that the operator $\mathcal{L}$ only operates on the *unprimed* coordinates. The function $g(\overline{r},\overline{r}')$ is called the "Green's function", and can be used to construct solutions to (24) for arbitrary current distributions as follows. Multiplying both sides of (25) by $J_x(\overline{r}')$ and integrating over primed coordinates gives

$$\iiint \mathcal{L}\, g(\overline{r},\overline{r}')\, J_x(\overline{r}')\, dV' = -\iiint \delta(\overline{r} - \overline{r}') J_x(\overline{r}')\, dV'$$

where the limits of integration encompass all space. The right hand side can be recognized as $-J_x(\overline{r})$ by the integral property of the delta function. Since $\mathcal{L}$ operates on unprimed coordinates, it can be pulled outside of the integral on the left hand side, giving

$$\mathcal{L} \iiint g(\overline{r},\overline{r}') J_x(\overline{r}')\, dV' = -J_x(\overline{r})$$

Comparing with (24) we identify a solution for A_x as

$$A_x(\overline{r}) = \mu \iiint g(\overline{r},\overline{r}')\, J_x(\overline{r}')\, dV'$$

and since all cartesian components satisfy the same form of equation, the general vector solution to (9a) is

$$\overline{A}(\overline{r}) = \mu \iiint g(\overline{r},\overline{r}') \overline{J}(\overline{r}')\, dV' \tag{26}$$

with a similar result for the vector potential $\overline{F}$. Physically, the Green's function is the impulse response of the system. The total response due to some arbitrary source distribution can then be found by expressing the source as a collection of impulses and adding up all of the individual responses. The result (26) is the mathematical statement of this superposition process.

This is a *particular* solution of (9a). A more general solution would also include the *complementary* solutions $\overline{A}_c$, which are solutions to the homogeneous equation

$$\mathcal{L}\, \overline{A}_c = 0$$

The general solution is thus

$$\overline{A}(\overline{r},t) = \overline{A}_c(\overline{r},t) + \mu \iiint g(\overline{r},\overline{r}')\overline{J}(\overline{r}')\,dV' \tag{27}$$

Equivalently, we could choose to modify the Green's function in the same way, adding to the particular solution of (25) solutions of the complementary equation $\mathcal{L}\,g_c = 0$. Such modifications are important in boundary value problems where the fields must satisfy specified boundary conditions. The complementary solutions will have unknown constants which are adjusted to satisfy these boundary conditions. Physically, a Green's function which satisfies the boundary conditions for a certain problem describes the impulse response of the system *including* the response due to secondary sources, such as induced conduction or polarization currents on material objects. If such a Green's function can be found and inserted into (26), then the resulting fields will automatically satisfy the same boundary conditions.

For problems involving radiation from current distributions in unbounded media, ie. with no conductors or dielectric materials present, the only boundary conditions to be satisfied are the boundary conditions at infinity, or radiation conditions. These unbounded Green's functions are generally easy to find, as we will see next. Unfortunately, modified Green's functions satisfying problem-specific boundary conditions are much more difficult to find in general.

3D Green's function for unbounded media

For unbounded media the solution to (25) is relatively straightforward. For simplicity, first consider the delta function to be centered at the origin. The Green's function will then have spherical symmetry and is a solution to

$$\left[\nabla^2 + k^2\right] g(r) = -\delta(\overline{r}) = -\frac{\delta(r)}{4\pi r^2} \tag{28}$$

Note the representation for the three dimensional delta function $\delta(\overline{r})$, which is found from the basic property $\iiint \delta(\overline{r})dV = 1$. For observation points away from the source singularity the Green's function satisfies the homogeneous wave equation and is given by

$$g(r) = A\frac{e^{-jkr}}{r}$$

where the "advanced" wave travelling in the $-\hat{r}$ direction is rejected on physical grounds (mathematically expressed using the radiation conditions). The unknown amplitude A is determined by inserting this form for $g(r)$ into (28) and integrating over a vanishingly small sphere centered at the origin

$$\lim_{a\to 0} \iiint_{\substack{\text{sphere}\\ \text{of radius } a}} \left[\nabla^2 + k^2\right] A\frac{e^{-jkr}}{r}\,dV = -1$$

The integral over the second term goes to zero since $dV \propto r^2$. Using the divergence theorem the first term is reduced to a surface integral, giving

$$\lim_{a\to 0} 4\pi a^2 \left.\frac{\partial g(r)}{\partial r}\right|_{r=a} = -1$$

which yields $A = 1/4\pi$. Shifting coordinate so that the origin is at $\overline{r}'$, we can generalize the result for a source at $\overline{r}'$ to give

$$g(\overline{r}, \overline{r}') = \frac{e^{-jk|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|} \tag{29}$$

The general solutions for the vector potentials are then given by

$$\overline{A}(\overline{r}) = \frac{\mu}{4\pi} \iiint \overline{J}(\overline{r}') \frac{e^{-jkR}}{R} dV' \tag{30a}$$

$$\overline{F}(\overline{r}) = \frac{\epsilon}{4\pi} \iiint \overline{M}(\overline{r}') \frac{e^{-jkR}}{R} dV' \tag{30b}$$

where $R = |\overline{r} - \overline{r}'|$. This is a formal (particular) solution to Maxwell's equations in unbounded media, and is an important result in elementary antenna theory since many problems can be reduced to a set of equivalent volume currents flowing in free-space.

2D Green's function for unbounded media

In two dimensions the Green's function in cylindrical coordinates satisfies

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + k^2\right] g(\rho) = -\delta(\bar{\rho}) = -\frac{\delta(\rho)}{2\pi\rho} \tag{31}$$

where the source is positioned at the origin for convenience so that g will possess azimuthal symmetry. Again, the representation for $\delta(\bar{\rho})$ is found from the integral property $\iint \delta(\bar{\rho})dS = 1$. For $\rho \neq 0$ this is a form of Bessel's equation with the general solution

$$g(\rho) = AJ_0(k\rho) + BN_0(k\rho)$$

where J_0 and N_0 are zero-order Bessel's functions of the first and second kind, respectively. However, for wave problems it is more convenient to use Hankel functions, which are linear combinations of of J_0 and N_0, defined by

$$H_0^{(1)}(x) = J_0(x) + jN_0(x) \tag{32a}$$

$$H_0^{(2)}(x) = J_0(x) - jN_0(x) \tag{32b}$$

These have the following asymptotic expansions for $x \gg 1$,

$$H_0^{(1)}(x) \rightarrow \sqrt{\frac{2}{\pi x}} e^{j(x-\pi/4)} \tag{33a}$$

$$H_0^{(2)}(x) \rightarrow \sqrt{\frac{2}{\pi x}} e^{-j(x-\pi/4)} \tag{33b}$$

which obviously have the desirable form of propagating waves; for our purpose, this is the only significant motivation for introducing the Hankel functions. In the present case only outward propagating waves are expected physically so

$$g(\rho) = AH_0^{(2)}(k\rho)$$

To find A we again integrate (31) around a small cylinder of radius a centered at the origin and let the radius go to zero,

$$\lim_{a\to 0}\int_0^{2\pi}\int_0^a\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+k^2\right]g(\rho)\,\rho\,d\rho\,d\phi=-1$$

Using the small argument approximation for the Hankel function

$$H_0^{(2)}(k\rho)\approx\frac{-2\jmath}{\pi}\ln(k\rho)\qquad\rho\ll 1$$

Using the same procedure as for the 3D Green's function, we find that $A=-\jmath/4$, so the Green's function is

$$g(\rho)=\frac{-\jmath}{4}H_0^{(2)}(k\rho)$$

for a source at the origin. Generalizing to a line source at $\bar{\rho}'$ we have the final result

$$g(\bar{\rho},\bar{\rho}')=\frac{-\jmath}{4}H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\tag{34}$$

General expressions for the vector potentials are then

$$A(\bar{\rho})=\frac{\mu}{4\jmath}\iint\overline{J}(\bar{\rho}')H_0^{(2)}(kR)\,dS'\tag{35a}$$

$$F(\bar{\rho})=\frac{\epsilon}{4\jmath}\iint\overline{M}(\bar{\rho}')H_0^{(2)}(kR)\,dS'\tag{35b}$$

where

$$R\equiv|\bar{\rho}-\bar{\rho}'|=\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')\right]^{1/2}$$

An alternate representation for the 2D Green's fuction can be obtained from our 3D result by noting that sources in two-dimensions are independent of z (infinite extent in the $\hat{z}$ direction) and consequently produce fields with no z dependence, so taking the observation point at $z=0$ in (30a) gives

$$\overline{A}(\rho)=\frac{\mu}{4\pi}\iint\overline{J}(\rho')\,dS'\int_{-\infty}^{\infty}\frac{e^{-\jmath k\sqrt{R^2+z'^2}}}{\sqrt{R^2+z'^2}}\,dz'$$

Comparing with (35a) we find an integral representation for the Hankel function

$$H_0^{(2)}(kR)=\frac{\jmath}{\pi}\int_{-\infty}^{\infty}\frac{e^{-\jmath k\sqrt{R^2+z'^2}}}{\sqrt{R^2+z'^2}}\,dz'\tag{36}$$

which is sometimes useful.

2 TE AND TM FIELD DECOMPOSITION

The integral expressions for the auxiliary potentials $(\overline{A},\overline{F})$ in (30) provide a *particular* solution to Maxwell's equations for the fields produced by a set of known sources in an unbounded and homogeneous region. A more general solution can be constructed by adding the *complementary*, or source-free solutions of Maxwell's equations. These are useful for inhomogeneous boundary-value or scattering problems where the fields must satisfy certain boundary conditions.

The source-free fields in a homogeneous region containing simple media satisfy

$$\nabla \cdot \overline{H} = 0 \qquad \nabla \times \overline{H} = \jmath\omega\epsilon\overline{E} \tag{37a}$$
$$\nabla \cdot \overline{E} = 0 \qquad \nabla \times \overline{E} = -\jmath\omega\mu\overline{H} \tag{37b}$$

or, equivalently, the vector wave equations

$$\nabla \times \nabla \times \overline{E} - k^2\overline{E} = \nabla^2\overline{E} + k^2\overline{E} = 0 \tag{38a}$$
$$\nabla \times \nabla \times \overline{H} - k^2\overline{H} = \nabla^2\overline{H} + k^2\overline{H} = 0 \tag{38b}$$

An important observation can be drawn immediately. Since the fields are divergenceless, only two of the three components of each vector field are independent; given two components of $\overline{E}$, for example, we can determine the third from $\nabla \cdot \overline{E}$=0. Furthermore, once we know $\overline{E}$, the field $\overline{H}$ is completely specified by $\overline{E}$ through Maxwell's equations according the the Helmoholtz theorem. Therefore, any arbitrary vector solution of the source-free Maxwell's equations can be completely specified by *two* independent scalar functions. We will contrive to make both of these scalar functions solutions of the scalar Helmholtz equation, for which there are well known solutions in commonly used coordinate systems.

The requirement $\nabla \cdot \overline{H} = 0$ allows us to write $\overline{H}$ as the curl of a vector function, which leads to a solution of the form

$$\overline{H} = \jmath\omega\epsilon\nabla \times \overline{\Pi}_e \quad \rightarrow \quad \overline{E} = \nabla \times \nabla \times \overline{\Pi}_e \tag{39}$$

where we have chosen the constants to match our earlier definition of the Hertz vector. Similarly the zero divergence of $\overline{E}$ leads to a solution of the form

$$\overline{E} = -\jmath\omega\mu\nabla \times \overline{\Pi}_m \quad \rightarrow \quad \overline{H} = \nabla \times \nabla \times \overline{\Pi}_m \tag{40}$$

The most general source-free solution can therefore be written as

$$\overline{E}(\overline{r}) \;=\; \nabla \times \nabla \times \overline{\Pi}_e - \jmath\omega\mu\nabla \times \overline{\Pi}_m \tag{41a}$$
$$\overline{H}(\overline{r}) \;=\; \jmath\omega\epsilon\nabla \times \overline{\Pi}_e + \nabla \times \nabla \times \overline{\Pi}_m. \tag{41b}$$

From Maxwell's equations, the source-free Hertz vectors must satisfy

$$\nabla \times \nabla \times \overline{\Pi}_e - k^2\overline{\Pi}_e \;=\; \nabla\phi_e \tag{42a}$$
$$\nabla \times \nabla \times \overline{\Pi}_m - k^2\overline{\Pi}_m \;=\; \nabla\phi_m \tag{42b}$$

where ϕ_e and ϕ_m are arbitrary scalar functions. As we saw earlier, there is a great deal of flexibility in choosing the scalar functions $\phi_{e/m}$ so as to simplify the resulting equations for the vector potentials. For example, choosing a Lorentz-like gauge $\nabla \cdot \Pi = \phi$ gives

$$\nabla^2\overline{\Pi}_e + k^2\overline{\Pi}_e \;=\; 0 \tag{43a}$$
$$\nabla^2\overline{\Pi}_m + k^2\overline{\Pi}_m \;=\; 0 \tag{43b}$$

This is not the only choice that will be useful to us, but it is an important one for problems in rectangular and cylindrical coordinates. For example, in rectangular coordinates, each components of the Hertz potentials must satisfy

$$\nabla^2\varphi + k^2\varphi = 0 \tag{44}$$

The $\hat{z}$ components of the potentials in cylindrical coordinates also satisfies this equation, which is apparent from (A.28).

At this point it may appear that we haven't done anything useful. We have introduced a set of vector potentials which turn out to satisfy the exact same equations as the original fields in (38)! However, it does turn out that the new potentials have an advantage, which derives from the representation (41). This representation allows us to generate a complete solution in terms of a *single* component of $\overline{\Pi}_e$ and the corresponding component of $\overline{\Pi}_m$. We pointed out earlier that only two scalar functions are required: these are the two we seek. By the proper choice of gauge, as above, we can make these scalar solutions satisfy the scalar Helmholtz equation. This representation is convenient for constructing solutions that satisfy common boundary conditions, as we will see later.

Consider the case when

$$\overline{\Pi}_e = \overline{c}\varphi \qquad \overline{\Pi}_m = 0 \tag{45}$$

where $\overline{c}$ is a constant vector, sometimes called the "pilot vector", and φ is a scalar function. From (41), and using (A.46) and the fact that $\nabla \times \overline{c} = 0$, we get

$$\overline{H} = \jmath\omega\epsilon\,\overline{c} \times \nabla\varphi \qquad \overline{E} = \nabla \times \nabla \times (\overline{c}\varphi) \tag{46}$$

The $\overline{H}$ fields in this case are always transverse to the pilot vector. For this reason, the fields are called *transverse magnetic* (TM) fields. Note that the *electric* Hertz vector generates a TM field, or one with only an electric field component in the direction of the pilot vector.

Similarly, if

$$\overline{\Pi}_e = 0 \qquad \overline{\Pi}_m = \overline{c}\varphi \tag{47}$$

then the fields are given by

$$\overline{E} = -\jmath\omega\mu\,\overline{c} \times \nabla\varphi \qquad \overline{H} = \nabla \times \nabla \times (\overline{c}\varphi) \tag{48}$$

The $\overline{E}$ fields in this case are always transverse to the pilot vector. For this reason, the fields are called *transverse electric* (TE) fields. A *magetic* Hertz vector generates a TE field.

With the proper choice of gauge and/or pilot vector, the scalar functions φ are solutions to the scalar Helmholtz equation. In general the TE and TM fields are subject to different boundary conditions, and therefore the scalar functions must be different in each case. An arbitrary solution to the source free Maxwell's equations can then be expressed as a superposition of the TE and TM fields,

$$\overline{E}(\overline{r}) = -\jmath\omega\mu\nabla \times \left(\overline{c}\varphi^{\mathrm{TE}}\right) + \nabla \times \nabla \times \left(\overline{c}\varphi^{\mathrm{TM}}\right) \tag{49a}$$

$$\overline{H}(\overline{r}) = \jmath\omega\epsilon\nabla \times \left(\overline{c}\varphi^{\mathrm{TM}}\right) + \nabla \times \nabla \times \left(\overline{c}\varphi^{\mathrm{TE}}\right). \tag{49b}$$

In most boundary value problems the solutions φ^{TE} or φ^{TM} will be expressed as series expansions in an infinite set of eigenmodes appropriate to the coordinate system.

Note that we have not *rigorously* proved that an arbitrary field can be expressed with a single choice for the pilot vector $\overline{c}$. A more detailed discussion and proof can be found in [1]. However, a few moments of reflection on (49) should convince you that the two scalar functions and any $\overline{c}$ can generate any arbitrary field. Remember that the divergenceless character of the fields is already built into this representation.

We have already seen that in rectangular coordinates, $\overline{c}$ can be any one of the cartesian components $\hat{x}$, $\hat{y}$, or $\hat{z}$; the choice will be dictated by the type of boundary conditions to be satisfied. In cylindrical coordinates, only the $\hat{z}$ component of the Hertz vectors will satisfy (44). In spherical coordinates, it turns out that a gauge can be found for which the choice $\overline{c} = \overline{r} = r\hat{r}$ leads to (44). Therefore, the possible choices for the pilot vector are:

- Rectangular coordinates: $\overline{c} = \hat{x}$, $\hat{y}$, or $\hat{z}$
- Cylindrical coordinates: $\overline{c} = \hat{z}$
- Spherical coordinates: $\overline{c} = \overline{r} = r\hat{r}$

3 HANSEN VECTOR WAVEFUNCTIONS

The TE and TM field decomposition is often written using the notation

$$\overline{M} = \nabla \times (\overline{c}\varphi) \qquad (50a)$$

$$\overline{N} = \frac{1}{k}\nabla \times \nabla \times (\overline{c}\varphi) = \frac{1}{k}\nabla \times \overline{M} \qquad (50b)$$

As mentioned, a typicaly boundary value problem will yield and infinite set of scalar eigenfunctions which satisfy the Helmholtz equation for a given set of boundary conditions. Therefore there will be an infinite set of vector wavefunctions $\overline{M}$ and N, and the source-free field can be represented as

$$\overline{E} = \sum_n \left[a_n \overline{M}_n + b_n \overline{N}_n\right] \qquad (51)$$

From Maxwell's equations we get

$$\overline{H} = \frac{-k}{\jmath\omega\mu} \sum_n \left[a_n \overline{N}_n + b_n \overline{M}_n\right] \qquad (52)$$

This representation was first introduced by Hansen [2]. Since the fields must satisfy (38), we find the additional property

$$\nabla \times \overline{N} = k\overline{M} \quad \text{or} \quad \overline{M} = \frac{1}{k}\nabla \times \overline{N} \qquad (53)$$

We can think of the vector wavefunctions as the linearly independent characteristic solutions, or eigenmodes, of the vector wave equation. These functions will also turn out to satisfy useful orthogonality relations which facilitate problem solving.

REFERENCES

[1] P.M. Morse and H.Fechbach, *Methods of Theoretical Physics*, vol.2, Chapter 13. McGraw-Hill: New York, 1953.

[2] W.W. Hansen, "A new type of expansion in radiation problems", *Phys. Rev.*, vol. 47, Jan 15, 1935, pp. 139-143.