

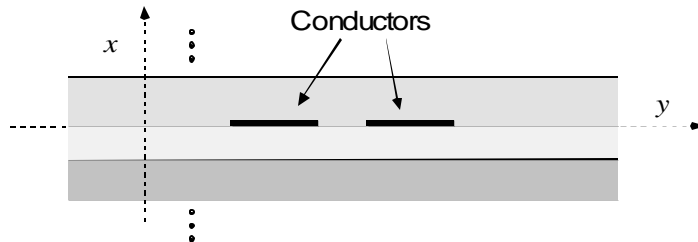
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# ANALYSIS OF PLANAR TRANSMISSION-LINE STRUCTURES

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## 1 GENERAL CONSIDERATIONS

Transmission-line or waveguiding structures used in high frequency microelectronic systems are frequently fabricated by photolithographic techniques on multilayered dielectric or semiconducting substrates. A representative example is shown in figure 1. Although the transmission-lines may have simple geometrical shapes, analysis is complicated by the inhomogeneous nature of the problem. Fortunately, there are many common features of such problems which lend themselves to a simple unified method of attack. We will focus on analysis of transmission-lines which are uniform along the  $z$ -direction, and hence can be reduced to analysis of a two-dimensional cross section. We will also assume infinitesimally thin perfect conductors for convenience. Although this rules out rigorous analysis of conductor losses, these can be approximated if needed by familiar perturbation methods.



**Figure 1** Example of a multiconductor planar transmission-line system in a multi-layered dielectric medium.

The dynamic fields in such multi-layer media are hybrid modes that can be expressed as a superposition of TE and TM modes. Within each individual dielectric layer, the TE and TM fields are generated by solutions of the scalar Helmholtz equation with the appropriate boundary conditions. Assuming wave propagation in the form of  $e^{-j\beta z}$ , the scalar Helmholtz equation is

$$[\nabla_t^2 + k_{ci}^2] \begin{Bmatrix} \psi_i^e \\ \psi_i^h \end{Bmatrix} = 0 \quad (1)$$

where

$$k_{ci}^2 = k_i^2 - \beta^2 = \omega^2 \mu_0 \epsilon_i - \beta^2$$

The index  $i$  denotes the  $i$ th dielectric region. The TM-to- $\hat{z}$  and TE-to- $\hat{z}$  fields are then generated by  $\psi^e$  and  $\psi^h$ , respectively, with the  $z$ -components of the fields expressed as

$$E_{zi} = k_{ci}^2 \psi_i^e(x, y) e^{-j\beta z} \quad (2a)$$

$$H_{zi} = k_{ci}^2 \psi_i^h(x, y) e^{-j\beta z} \quad (2b)$$

and the transverse fields are determined from

$$\overline{E}_{ti} = -j\beta \nabla_t \psi_i^e + j\omega\mu \hat{z} \times \nabla_t \psi_i^h \quad (3a)$$

$$\overline{H}_{ti} = -j\omega\epsilon_i \hat{z} \times \nabla_t \psi_i^e - j\beta \nabla_t \psi_i^h \quad (3b)$$

The scalar solutions  $\psi^e$  and  $\psi^h$  must therefore satisfy the following boundary conditions:

$$\begin{aligned} \psi_i^e &= 0 & \text{on PEC} & & \frac{\partial \psi_i^h}{\partial n} &= 0 & \text{on PEC} \\ \frac{\partial \psi_i^e}{\partial n} &= 0 & \text{on PMC} & & \psi_i^h &= 0 & \text{on PMC} \end{aligned} \quad (4)$$

If the currents on the conductors were known, analysis of the inhomogeneous system could then proceed by constructing field solutions in each dielectric region, and enforcing boundary conditions across the interfaces to generate a so-called ‘‘characteristic equation’’ for the unknown propagation constant, and subsequently relating the unknown field amplitudes in each region. However, the currents are rarely known *a priori*. One must then resort to approximation techniques or numerical methods.

## 2 SPACE DOMAIN INTEGRAL EQUATION FORMULATION

If we can find a Green’s function for the problem, we can set up an integral equation for the unknown current distribution as follows. Assuming infinitesimally thin PEC strips in the  $x = 0$  plane as shown in figure 1, the field is written as

$$\overline{E}(x, y) = \int_{strip} \overline{\overline{G}}(x, y|0, y') \cdot \overline{J}_s(y') dy' \quad (5)$$

Note that the Green’s function here is an implicit function of the propagation constant,  $\beta$ . The tangential field must vanish at the conductors, so  $\hat{x} \times \overline{E}(0, y) = 0$  when  $y$  lies on a conducting strip. This leads to the coupled integral equations for the unknown surface currents

$$E_y(0, y) = \int G_{yy}(y|y') J_y(y') dy' + \int G_{yz}(y|y') J_z(y') dy' = 0 \quad (6a)$$

when  $y$  on metal

$$E_z(0, y) = \int G_{zy}(y|y') J_y(y') dy' + \int G_{zz}(y|y') J_z(y') dy' = 0 \quad (6b)$$

where it is understood that the Green’s functions are evaluated on the conductors at  $x = x' = 0$ . Following the Method of Moments procedure, we expand the currents in a set of

known basis functions as

$$J_y(y) = \sum_{i=1}^{N_y} a_{yi} \xi_{yi}(y) \quad J_z(y) = \sum_{j=1}^{N_z} a_{zj} \xi_{zj}(y) \quad (7)$$

It is important to choose basis functions  $\xi(y)$  that satisfy the boundary conditions, ie. the functions must be non-zero only over the conducting part of the interface. Substituting (7) into (6) gives

$$\sum_{i=1}^{N_y} a_{yi} \int G_{yy}(y|y') \xi_{yi}(y') dy' + \sum_{j=1}^{N_z} a_{zj} \int G_{yz}(y|y') \xi_{zj}(y') dy' = 0 \quad (8a)$$

*y* on metal

$$\sum_{i=1}^{N_y} a_{yi} \int G_{zy}(y|y') \xi_{yi}(y') dy' + \sum_{j=1}^{N_z} a_{zj} \int G_{zz}(y|y') \xi_{zj}(y') dy' = 0 \quad (8b)$$

where the integrals are carried out over the conductors at  $x = 0$ . We can generate a matrix equation using the basis set as weighting functions (Galerkin's method). Multiplying (8a) by  $\xi_{yp}$  and (8b) by  $\xi_{zq}$  and integrating over the strips gives

$$\sum_{i=1}^{N_y} a_{yi} K_{pi}^{yy}(\beta) + \sum_{j=1}^{N_z} a_{zj} K_{pj}^{yz}(\beta) = 0 \quad p = 1 \dots N_y \quad (9a)$$

$$\sum_{i=1}^{N_y} a_{yi} K_{qi}^{zy}(\beta) + \sum_{j=1}^{N_z} a_{zj} K_{qj}^{zz}(\beta) = 0 \quad q = 1 \dots N_z \quad (9b)$$

where

$$K_{pi}^{yy}(\beta) = \iint \xi_{yp}(y) G_{yy}(y|y') \xi_{yi}(y') dy dy' \quad p = 1 \dots N_y, \quad i = 1 \dots N_y$$

$$K_{pj}^{yz}(\beta) = \iint \xi_{yp}(y) G_{yz}(y|y') \xi_{zj}(y') dy dy' \quad p = 1 \dots N_y, \quad j = 1 \dots N_z$$

$$K_{qi}^{zy}(\beta) = \iint \xi_{zq}(y) G_{zy}(y|y') \xi_{yi}(y') dy dy' \quad q = 1 \dots N_z, \quad i = 1 \dots N_y$$

$$K_{qj}^{zz}(\beta) = \iint \xi_{zq}(y) G_{zz}(y|y') \xi_{zj}(y') dy dy' \quad q = 1 \dots N_z, \quad j = 1 \dots N_z$$

The integrals are taken over the conducting strips. The dependence of the matrix elements on the propagation constant has also been explicitly shown. For there to be a nontrivial solution for the expansion coefficients, the determinant of the coefficient matrix must vanish,

$$\begin{vmatrix} \overline{\overline{K}}^{yy}(\beta) & \overline{\overline{K}}^{yz}(\beta) \\ \overline{\overline{K}}^{zy}(\beta) & \overline{\overline{K}}^{zz}(\beta) \end{vmatrix} = 0 \quad (10)$$

This is a characteristic equation for the unknown propagation constant, which can be solved using non-linear root-finding algorithms. At any frequency there can be more than one

value for the propagation constant which will satisfy (10), corresponding to allowed modes of the transmission-line structure. Usually the lowest order solution is desired. Once the propagation constant is found, the unknown expansion coefficients  $a_{yi}$  and  $a_{zj}$  can be found from (9). This is done by assigning the value of unity to one coefficient, which leaves an overdetermined set of linear equations which can be solved for a “best fit” by simple least-squares or linear regression algorithms. Once both the propagation constant and surface currents are known, all the fields can be computed from (5).

In practice, the above result is of limited use. Even when suitable space-domain Green’s functions can be found, the procedure requires repeated evaluation of two-dimensional integrals in a nonlinear root-finding algorithm, which can be prohibitively time consuming. The singularity of the Green’s function at  $y = y'$  can also lead to numerical difficulties. The computational efficiency can be improved somewhat in those cases where the current distribution can be accurately guessed *a priori*, such that only one basis function is required, and/or by using point-matching techniques (Dirac delta weighting functions) to reduce the two-dimensional integrals to one-dimensional integrals. Techniques for approximating the Green’s function by elementary functions using the method of “complex images” have also been developed for planar layered media [?], which can speed the procedure greatly if the resulting integrals can be evaluated analytically.

### 3 SPECTRAL DOMAIN ANALYSIS

Application of the Fourier transform can solve many of the difficulties with the space domain integral equation method, and leads to the so-called “Spectral Domain Method” (SDM). This technique was first developed for shielded microstrip lines by Itoh and Mittra [1], and has since been extended to a large number of transmission-line structures. The SDM is essentially a procedure for computing the Green’s function for the problem by Fourier decomposition, reducing the problem to that of solving algebraic equations. Provided that the basis set for the Moment Method solution can be transformed analytically, it reduces the computation burden to the evaluation of integrals or series in one dimension. The key requirements of the Spectral Domain Method (SDM) are the existence of planar dielectric regions which are uniform in lateral extent, and infinitesimally thin perfect conductors which lie along a dielectric interface. Some examples are shown in figure 2. Although these structures are shown in finite enclosures, the dimensions of the enclosure can be made infinite in one or more directions, and each boundary can be chosen as PEC or PMC independently. Since each dielectric region is homogeneous in the  $\hat{y}$ -direction with a uniform length  $b$ , the  $y$ -dependence of the fields in each region can be written as a Fourier series

$$\psi(x, y) = \sum_{n=-\infty}^{\infty} \tilde{\psi}(x, \alpha_n) e^{-j\alpha_n y} \quad (11)$$

where  $\alpha_n = 2n\pi/b$  or  $\alpha_n = (2n - 1)\pi/b$ , depending on the boundary conditions. If the enclosure width  $b$  is extended to infinity, the Fourier series would become a Fourier integral. The function  $\psi$  is just the Fourier transform (in  $y$ ) of the spatial fields

$$\tilde{\psi}(x, \alpha_n) = \frac{1}{b} \int_{-b/2}^{b/2} \psi(x, y) e^{j\alpha_n y} dy \quad (12)$$

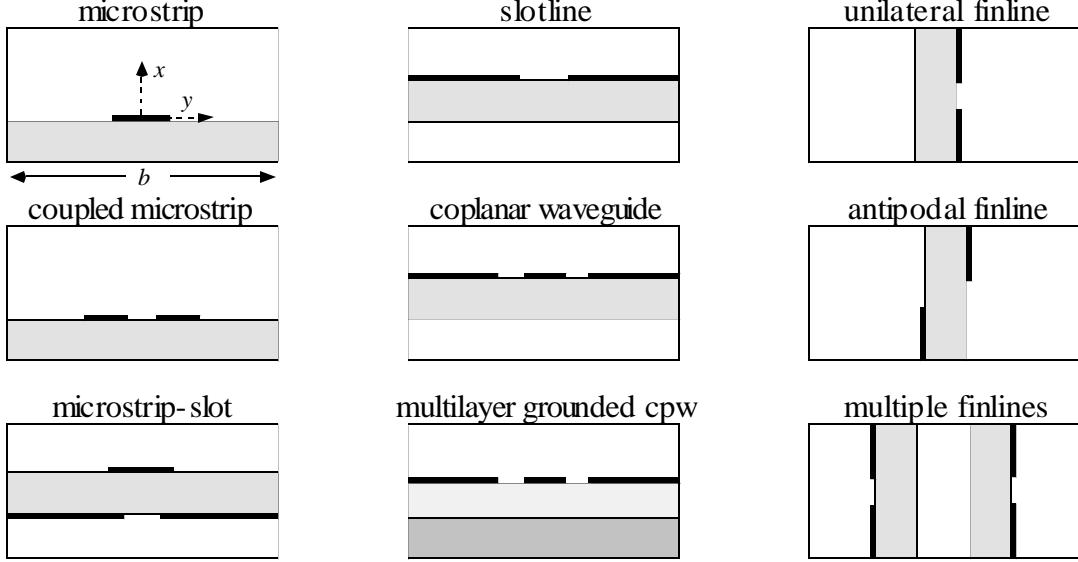


Figure 2 Examples of planar transmission structures that can be treated by the Spectral Domain Method.

Inserting (11) into (1), we find that the fields in the Fourier domain, or “spectral domain”, must satisfy the wave equation

$$\left[ \frac{\partial^2}{\partial x^2} - \Gamma_{ni}^2 \right] \left\{ \begin{array}{l} \tilde{\psi}_{ni}^e \\ \tilde{\psi}_{ni}^h \end{array} \right\} = 0 \quad (13)$$

where

$$\Gamma_{ni}^2 = \alpha_n^2 - k_{ci}^2 = \alpha_n^2 + \beta^2 - \omega^2 \mu \epsilon_i$$

and the index  $i$  runs over all the layers. The problem then reduces to a set of 1-dimensional wave equations in the  $\hat{x}$ -direction. Expressions for the fields in each region are then found, which must be made consistent with the boundary conditions at the interfaces. The field matching across boundaries is facilitated by the Fourier representation of the fields, since the spatial harmonics are mutually orthogonal. For convenience, the field expressions in the spectral domain for lossless propagating modes with  $\gamma = j\beta$  are summarized here:

$$\tilde{E}_z(x, \alpha_n) = k_c^2 \tilde{\psi}_n^e \quad (14a)$$

$$\tilde{H}_z(x, \alpha_n) = k_c^2 \tilde{\psi}_n^h \quad (14b)$$

$$\tilde{E}_x(x, \alpha_n) = -j\beta \frac{\partial \tilde{\psi}_n^e}{\partial x} - \omega \mu \alpha_n \tilde{\psi}_n^h \quad (14c)$$

$$\tilde{E}_y(x, \alpha_n) = -\beta \alpha_n \tilde{\psi}_n^e + j\omega \mu \frac{\partial \tilde{\psi}_n^h}{\partial x} \quad (14d)$$

$$\tilde{H}_x(x, \alpha_n) = \omega \epsilon \alpha_n \tilde{\psi}_n^e - j\beta \frac{\partial \tilde{\psi}_n^h}{\partial x} \quad (14e)$$

$$\tilde{H}_y(x, \alpha_n) = -j\omega \epsilon \frac{\partial \tilde{\psi}_n^e}{\partial x} - \beta \alpha_n \tilde{\psi}_n^h \quad (14f)$$

### 3.1 Analysis of Shielded Microstrip

Further description of the method is probably best done in the context of an actual problem. We consider the simple case of a shielded microstrip line as shown in fig. 3.

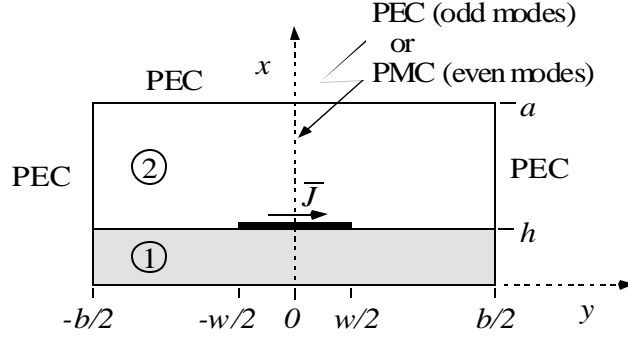


Figure 3 Geometry for SDM analysis of microstrip.

We will consider only modes with even symmetry about  $y = 0$ , which includes the dominant modes of the microstrip. In this case,

$$\psi(x, y) = \psi(x, -y) \quad (15)$$

If the boundaries at  $y = \pm b/2$  are PEC as shown in fig. 3, then this implies that

$$\psi(x, \frac{b}{2} + y) = -\psi(x, \frac{b}{2} - y) = -\psi(x, y - \frac{b}{2}) \quad (16)$$

where the last equality follows from (15). Inserting (11) into (16) we find that the boundary conditions for even modes require

$$\alpha_n = \frac{(2n-1)\pi}{b} \quad (\text{even modes}) \quad (17)$$

The dielectric interfaces involving conductors will support surface currents that are not known *a priori*. We wish to relate these unknown conduction currents to the corresponding tangential electric fields at the interfaces, as in (6). Assuming a PEC enclosure, the TE and TM fields (to  $\hat{z}$ ) are generated by solution of (13) subject to (4)

Region-1	Region-2	
$\tilde{\psi}_{n1}^e = a_n \sinh \Gamma_{n1} x$	$\tilde{\psi}_{n2}^e = c_n \sinh \Gamma_{n2} (a - x)$	(18)
$\tilde{\psi}_{n1}^h = b_n \cosh \Gamma_{n1} x$	$\tilde{\psi}_{n2}^h = d_n \cosh \Gamma_{n2} (a - x)$	
$\Gamma_{n1} = \alpha_n^2 + \beta^2 - k_0^2 \epsilon_{r1}$	$\Gamma_{n2} = \alpha_n^2 + \beta^2 - k_0^2 \epsilon_{r2}$	

the unknown coefficients ( $a_n, b_n, c_n, d_n$ ) are found by enforcing the boundary conditions at the interface plane  $x = h$ . In the spectral domain, the boundary conditions are

$$\begin{aligned} \tilde{E}_{z1}(h, \alpha_n) &= \tilde{E}_{z2}(h, \alpha_n) & \tilde{H}_{z1}(h, \alpha_n) - \tilde{H}_{z2}(h, \alpha_n) &= \tilde{J}_{sy}(\alpha_n) \\ \tilde{E}_{y1}(h, \alpha_n) &= \tilde{E}_{y2}(h, \alpha_n) & \tilde{H}_{y1}(h, \alpha_n) - \tilde{H}_{y2}(h, \alpha_n) &= -\tilde{J}_{sz}(\alpha_n) \end{aligned} \quad (19)$$

Substituting (18) into (19) leads to a set of simultaneous algebraic equations for the unknown coefficients. With the following shorthand,

$$P_{ni} \equiv \frac{\beta \alpha_n}{k_{ci}^2} \quad Q_{ni} \equiv \frac{j\omega\mu\Gamma_{ni}}{k_{ci}^2} \quad R_{ni} \equiv \frac{j\epsilon_i\Gamma_{ni}}{k_{ci}^2}$$

$$\begin{aligned} \Delta P &\equiv P_{n2} - P_{n1} \\ \Delta Q &\equiv Q_{n1} \tanh \Gamma_{n1} h + Q_{n2} \tanh \Gamma_{n2} (a - h) \\ \Delta R &\equiv R_{n1} \coth \Gamma_{n1} h + R_{n2} \coth \Gamma_{n2} (a - h) \end{aligned}$$

and some tedious manipulation we find the following solutions for the fields

$$a_n = \frac{\tilde{J}_{sy} [Q_{n2} \Delta P \tanh \Gamma_{n2} (a - h) - P_{n2} \Delta Q] + \tilde{J}_{sz} \Delta Q}{k_{c1}^2 \sinh \Gamma_{n1} h (\Delta P \Delta P + \Delta Q \Delta R)} \quad (20a)$$

$$b_n = \frac{\tilde{J}_{sy} [Q_{n2} \Delta R \tanh \Gamma_{n2} (a - h) + P_{n2} \Delta P] - \tilde{J}_{sz} \Delta P}{k_{c1}^2 \cosh \Gamma_{n1} h (\Delta P \Delta P + \Delta Q \Delta R)} \quad (20b)$$

$$c_n = a_n \frac{k_{c1}^2 \sinh \Gamma_{n1} h}{k_{c2}^2 \sinh \Gamma_{n2} (a - h)} \quad (20c)$$

$$d_n = \frac{b_n k_{c1}^2 \cosh \Gamma_{n1} h - \tilde{J}_{sy}}{k_{c2}^2 \cosh \Gamma_{n2} (a - h)} \quad (20d)$$

Combining (14a), (14d), (18), and (20), we can construct equations relating the interface fields to the interface currents in the form of

$$\tilde{E}_y(h, \alpha_n) = \tilde{Z}_{yy}(\alpha_n) \tilde{J}_{sy}(\alpha_n) + \tilde{Z}_{yz}(\alpha_n) \tilde{J}_{sz}(\alpha_n) \quad (21a)$$

$$\tilde{E}_z(h, \alpha_n) = \tilde{Z}_{zy}(\alpha_n) \tilde{J}_{sy}(\alpha_n) + \tilde{Z}_{zz}(\alpha_n) \tilde{J}_{sz}(\alpha_n) \quad (21b)$$

By comparison with (6), we see that the ‘‘impedance’’ functions ( $Z_{yy}$ , *etc.*) are just the Fourier transforms of the space-domain Green’s function, evaluated on the current-carrying interface, and are:

$$\tilde{Z}_{yy}(\alpha_n) = \frac{(P_{n1}^2 + Q_{n1} \Delta R \tanh \Gamma_{n1} h) Q_{n2} \tanh \Gamma_{n2} (a - h) + P_{n2}^2 Q_{n1} \tanh \Gamma_{n1} h}{\Delta P \Delta P + \Delta Q \Delta R} \quad (22a)$$

$$\tilde{Z}_{yz}(\alpha_n) = \tilde{Z}_{zy}(\alpha_n) = -\frac{P_{n1} Q_{n2} \tanh \Gamma_{n2} (a - h) + P_{n2} Q_{n1} \tanh \Gamma_{n1} h}{\Delta P \Delta P + \Delta Q \Delta R} \quad (22b)$$

$$\tilde{Z}_{zz}(\alpha_n) = \frac{\Delta Q}{\Delta P \Delta P + \Delta Q \Delta R} \quad (22c)$$

Later we will see how to compute these functions in a more convenient way.

### 3.2 Solution for the Propagation Constant

In general, the SDM deals specifically with the type of formulation in (21). The advantage of SDM is the straightforward (though perhaps quite tedious) algebraic solution for the spectral-domain Green’s functions. Once these Green’s functions are found, the procedure

for manipulating (21) into a solution for the propagation constant then closely parallels the Method of Moments, where the unknown functions (either the tangential fields or currents at the interface) are expanded in terms of a known basis set, and a matrix equation is generated for the unknown expansion coefficients. For the microstrip case, it is appropriate to expand the unknown currents on the strip as

$$J_y(h, y) = \sum_{i=1}^{N_y} a_{yi} \xi_{yi}(y) \quad J_z(h, y) = \sum_{i=1}^{N_z} a_{zi} \xi_{zi}(y) \quad (23)$$

It is extremely important to choose basis functions  $\xi_i(y)$  that satisfy the boundary conditions, *i.e.* the functions must be non-zero only over the conducting part of the interface. These expressions are then Fourier transformed and inserted into (21). The unknown aperture fields are then eliminated by multiplying both sides of (21a) and (21b) with one of the basis functions and summing over all the spatial harmonics, making use of the fact that

$$\int_{-b/2}^{b/2} E_t(h, y) \xi_{tm}^*(y) dy = \sum_{n=-\infty}^{\infty} \tilde{E}_t(h, \alpha_n) \tilde{\xi}_{tm}^*(\alpha_n) = 0 \quad (24)$$

where the first equality follows from Parseval's identity, and the second equality follows from the fact that wherever  $E_t$  is non-zero,  $J_t$  (or  $\xi_t$ ) is zero, and vice-versa. The subscript  $t$  here denotes any tangential coordinate ( $y$  or  $z$ ). Following this procedure then gives the following matrix equation of the form (9), where the matrix elements are now given by

$$\begin{aligned} K_{pi}^{yy}(\beta) &= \sum_{n=-\infty}^{\infty} \tilde{\xi}_{yp}^*(\alpha_n) \tilde{Z}_{yy}(\alpha_n) \tilde{\xi}_{yi}(\alpha_n) & p = 1 \dots N_y, \quad i = 1 \dots N_y \\ K_{pj}^{yz}(\beta) &= \sum_{n=-\infty}^{\infty} \tilde{\xi}_{yp}^*(\alpha_n) \tilde{Z}_{yz}(\alpha_n) \tilde{\xi}_{zj}(\alpha_n) & p = 1 \dots N_y, \quad j = 1 \dots N_z \\ K_{qi}^{zy}(\beta) &= \sum_{n=-\infty}^{\infty} \tilde{\xi}_{zq}^*(\alpha_n) \tilde{Z}_{zy}(\alpha_n) \tilde{\xi}_{yi}(\alpha_n) & q = 1 \dots N_z, \quad i = 1 \dots N_y \\ K_{qj}^{zz}(\beta) &= \sum_{n=-\infty}^{\infty} \tilde{\xi}_{zq}^*(\alpha_n) \tilde{Z}_{zz}(\alpha_n) \tilde{\xi}_{zj}(\alpha_n) & q = 1 \dots N_z, \quad j = 1 \dots N_z \end{aligned}$$

The propagation constant is then found from the characteristic equation (10). Once the propagation constant is known, the expansion coefficients for the current are determined from (9), and then the fields can be computed using (14), (18), and (20). The computational procedure is therefore complete. Only the basis set for expansion of the unknown current distribution must be specified.

### *Dispersion in Microstrip*

One advantage of the SDM formulation when applied to commonly-used "quasi-TEM" transmission-lines is that relatively simple approximations to the current distribution on the line yield quite accurate numerical results for the propagation constant. As an example, we consider a simple two-term approximation to the current on the microstrip line of the

form

$$\xi_y(y) = \frac{y\sqrt{(w/2)^2 - y^2}}{(w/2)^2} \quad \xi_z(y) = \frac{w/2}{\sqrt{(w/2)^2 - y^2}} \quad (25)$$

So that the currents are represented by

$$J_{sy}(y) = a_y \xi_y(y) \quad J_{sz}(y) = a_z \xi_z(y) \quad (26)$$

The basis set in (25) can be easily transformed using (12) as

$$\tilde{\xi}_{ym} = \frac{J\pi}{b\alpha_n} J_2\left(\frac{w}{2}\alpha_n\right) \quad \tilde{\xi}_{zn} = \frac{w\pi}{2} J_0\left(\frac{w}{2}\alpha_n\right) \quad (27)$$

where the  $J_0(x)$  and  $J_2(x)$  are Bessel functions of the first kind, of order 0 and 2 respectively.

A *Mathematica* program was written to compute the propagation constant and effective permittivity. This program is included as an appendix. A representative calculation for a microstrip line is shown in fig. 4.

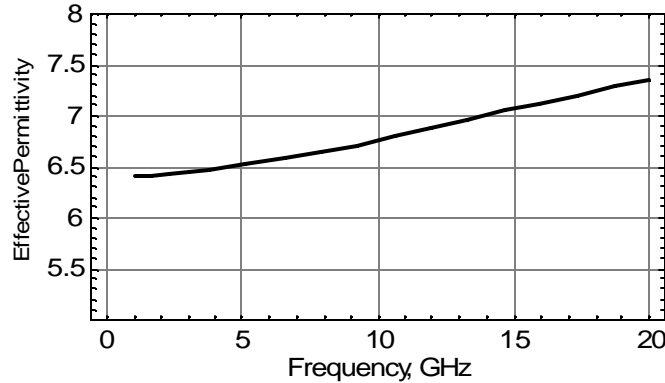


Figure 4 Effective permittivity of a 50 - microstrip line on  $h=25$  mil Alumina ( $\epsilon_r = 9.8$ ), with  $w/h = 1$ .

### 3.3 Characteristic Impedance

Characteristic impedance on a transmission line can be computed in one of three ways, using the *voltage-current*, *power-current* or *power-voltage* definitions; these are

$$Z_0 = V/I \quad Z_0 = 2P_{ave}/I^2 \quad \text{or} \quad Z_0 = V^2/(2P_{ave}) \quad (28)$$

where  $P_{ave} = VI^*/2$  is the time-averaged power flowing through a cross section of the structure. For a wave travelling on an ideal TEM transmission line, all three definitions give the same result. However, in a non-TEM structure there can be significant differences in these definitions, owing to the ambiguity in defining a unique voltage and/or current. In these cases, the “best” definition is frequently a matter of debate, ultimately decided by correlation with experimental data.

Most commonly used planar multi-layered structures are operated in a “quasi-TEM” mode where there is only a slight departure from true TEM behavior. These structures are characterized by conductor separations and dimensions that are small compared with a wavelength in the surrounding material, and hence the time-varying fields resemble the static fields. The three definitions for characteristic impedance then give quantitatively similar results, especially at low frequencies. However, the power-current and power-voltage definitions tend to be most useful, because the time-averaged power can be unambiguously computed from the fields as

$$\begin{aligned} P_{ave} &= \frac{1}{2} \text{Re} \iint (\overline{\mathbf{E}} \times \overline{\mathbf{H}}^*) \cdot \hat{z} \, dx \, dy \\ &= \frac{1}{2} \text{Re} \iint (E_x H_y^* - E_y H_x^*) \, dx \, dy \end{aligned} \quad (29)$$

Using Parseval’s identity, (29) is equivalent to

$$P_{ave} = \frac{b}{2} \text{Re} \sum_{n=-\infty}^{\infty} \int_0^a (\tilde{E}_{xn} \tilde{H}_{yn}^* - \tilde{E}_{yn} \tilde{H}_{xn}^*) \, dx \quad (30)$$

Substituting (14) into (30), and assuming that either  $\psi^e = 0$  or  $\psi^h = 0$  at the top and bottom boundary ( $x = 0, a$ ), it can be shown that

$$P_{ave} = \frac{\omega b \beta}{2} \sum_{n=-\infty}^{\infty} \int_0^a \left[ \epsilon \alpha_n^2 |\tilde{\psi}_n^e|^2 + \mu \alpha_n^2 |\tilde{\psi}_n^h|^2 + \epsilon \left| \frac{d\tilde{\psi}_n^e}{dx} \right|^2 + \mu \left| \frac{d\tilde{\psi}_n^h}{dx} \right|^2 \right] dx \quad (31a)$$

$$= \frac{\omega b \beta}{2} \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^L W_{n\ell} \quad (31b)$$

where  $L$  is the number of layers in the structure, and

$$W_{n\ell} = \int_{\ell} \left[ \epsilon_{\ell} \alpha_n^2 |\tilde{\psi}_{n\ell}^e(x)|^2 + \mu_{\ell} \alpha_n^2 |\tilde{\psi}_{n\ell}^h(x)|^2 + \epsilon_{\ell} \left| \frac{d\tilde{\psi}_{n\ell}^e}{dx} \right|^2 + \mu_{\ell} \left| \frac{d\tilde{\psi}_{n\ell}^h}{dx} \right|^2 \right] dx$$

This is a general result, applicable to all SDM problems. The only remaining step in the computation of impedance is the computation of either voltage or current. The particular geometry of the transmission-line structure may lend itself to the convenient computation of one or the other

### *Microstrip Characteristic Impedance*

In the microstrip geometry, the current along the top strip is most readily and unambiguously deduced from the SDM analysis; a knowledge of the current distribution on the strip (or an approximation to it) is in fact a consequence of the analysis. This suggests using the power-current definition for impedance, and in fact this definition tends to agree well with experiment. For the 2-term approximation described earlier, the total  $\hat{z}$ -directed current would be

$$I = \int_{-w/2}^{w/2} J_{sz}(y) \, dy = a_z \int_{-w/2}^{w/2} \xi_z(y) \, dy = \frac{\pi w}{2} \quad (32)$$

The next step is to compute the average power flow. In the case of the microstrip geometry, the fields from (18) can be inserted into (31) to evaluate the  $W_{nl}$ , but the unknown field amplitudes must first be determined. If the same two-term expansion is used as in (26), then (9) becomes

$$\begin{aligned} a_y K_{yy}(\beta) + a_z K_{yz}(\beta) &= 0 \\ a_y K_{zy}(\beta) + a_z K_{zz}(\beta) &= 0 \end{aligned} \quad (33)$$

Once  $\beta$  is found (as was done in fig. 4), the current expansion coefficients ( $a_y, a_z$ ) can be found by setting  $a_z = 1$  and solving for  $a_y$  as

$$a_y = -\frac{K_{yz} + K_{zz}}{K_{yy} + K_{zy}} \quad (34)$$

Knowledge of these expansion coefficients and the Fourier transforms (27) then allows us to compute the fields in (18) using the expressions (20).

The attached *Mathematica* file describes the computation of characteristic impedance using this procedure. A representative calculation for a microstrip line is shown in fig. 5.

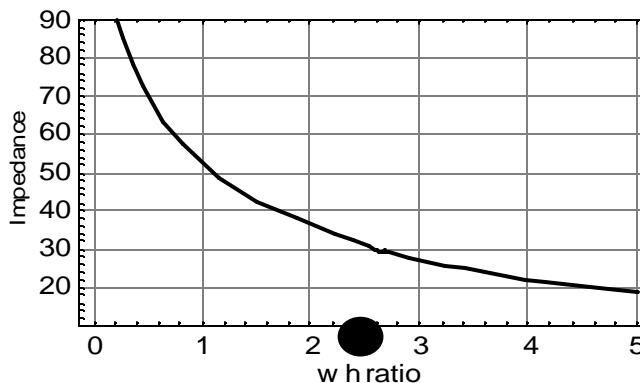


Figure 5 Characteristic impedance versus  $w/h$  ratio for a microstrip line on  $h=25$  mil Alumina ( $\epsilon_r = 9.8$ ), at a frequency of 1 GHz.

### 3.4 Equivalent Circuit Method for Computing Green's Functions

Itoh [2] has shown that the relationships between interfacial currents and electric fields in the spectral domain, such as (21) for the microstrip case, can be derived in a convenient and intuitive way using a simple equivalent transmission-line concept. This might be anticipated from the 1-dimensional wave equation (13) which governs the fields in the spectral domain. Note first that the fields can all be expressed in the general form

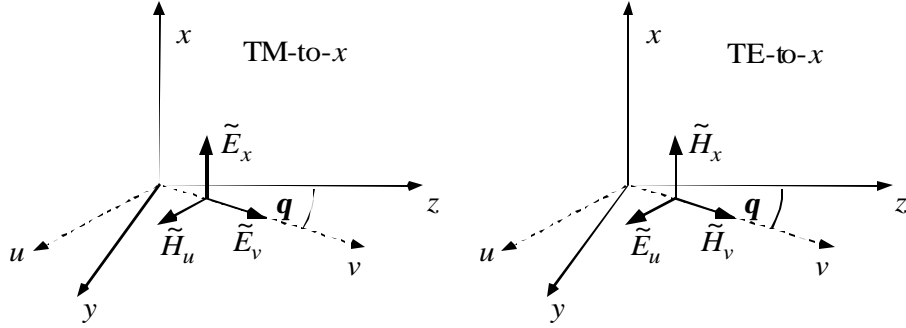
$$\bar{E}(x, y) e^{-j\beta z} = \sum_{n=-\infty}^{\infty} \tilde{E}_n(x) e^{-j(\alpha_n y + \beta z)} \quad (35)$$

which is a superposition of waves travelling in the  $y$ - $z$  plane. The  $n$ th term in this summation travels along the direction  $\theta$  from the  $z$ -axis, where

$$\cos \theta = \frac{\beta}{\alpha_n^2 + \beta^2} \quad \sin \theta = \frac{\alpha_n}{\alpha_n^2 + \beta^2} \quad (36)$$

Each such wave can be expressed as a superposition of fields TM-to- $x$  ( $\tilde{E}_v, \tilde{H}_u, \tilde{E}_x$ ) and TE-to- $x$  ( $\tilde{E}_u, \tilde{H}_v, \tilde{H}_x$ ), where the coordinates  $(u, v)$  are shown in fig. 6, and are related to  $(x, y)$  via

$$\begin{aligned} u &= y \cos \theta - z \sin \theta \\ v &= y \sin \theta + z \cos \theta \end{aligned} \quad (37)$$



**Figure 6** Transformation to  $(x, u, v)$  coordinates, and field components for the basic LSE and LSM modes.

This is the first of two key insights. The second is to recognize that a current element  $\tilde{J}_v$  couples only to the TM-to- $x$  fields, and a current element  $\tilde{J}_u$  couples only to the TE-to- $x$  fields. Since  $\tilde{J}_u$  and  $\tilde{J}_v$  both lie in the  $y$ - $z$  plane, then they can be represented as linear combinations of the currents  $\tilde{J}_y$  and  $\tilde{J}_z$  through the coordinate transformation (37). In other words, we can ultimately link  $\tilde{J}_u$  and  $\tilde{J}_v$  to the true interfacial currents. Recalling that the field components also obey a wave equation in the  $x$ -direction, we can now draw two equivalent transmission-line circuits, one for the TM-to- $x$  fields, and one for the TE-to- $x$  fields, as shown in fig. 7 for the case of the microstrip line.

In the equivalent circuits for the microstrip line, the interfacial currents on the conducting strips are represented as current sources at the junction of two transmission-lines, one representing the substrate (region-1) and the other representing the air above the microstrip (region-2). All the boundary conditions at the interfaces and the shield are incorporated into the equivalent circuits. The propagation constant in each equivalent transmission line is given by  $\Gamma_{ni}$  in (13). The “characteristic admittance” in each line is the wave admittance appropriate to the mode,

$$Y_{ni}^{\text{TM}} = \frac{\tilde{H}_u}{\tilde{E}_v} = \frac{j\omega\epsilon_i}{\Gamma_{ni}} \quad Y_{ni}^{\text{TE}} = \frac{\tilde{H}_v}{\tilde{E}_u} = \frac{\Gamma_{ni}}{j\omega\mu} \quad (38)$$

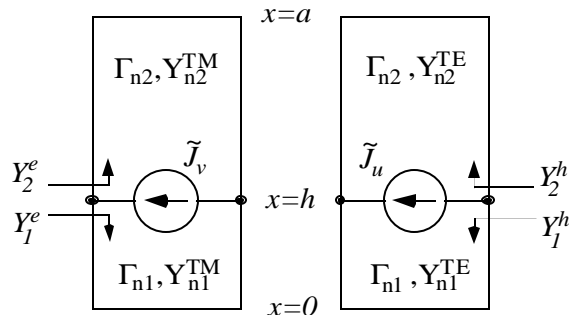


Figure 7 Equivalent transmission-line circuits for the microstrip problem of fig. 3.

We can then relate the tangential fields at the interface to the currents using transmission-line theory,

$$\tilde{E}_v(h, \alpha_n) = \tilde{Z}^e(\alpha_n) \tilde{J}_v(\alpha_n) \quad (39a)$$

$$\tilde{E}_u(h, \alpha_n) = \tilde{Z}^h(\alpha_n) \tilde{J}_u(\alpha_n) \quad (39b)$$

where  $\tilde{Z}^e$  and  $\tilde{Z}^h$  are the input impedances at  $x=d$ ; using the notation of fig. 7, these are

$$\tilde{Z}^e = \frac{1}{Y_1^e + Y_2^e} \quad \tilde{Z}^h = \frac{1}{Y_1^h + Y_2^h} \quad (40)$$

where

$$Y_1^e = Y_{n1}^{\text{TM}} \coth \Gamma_{n1} h \quad Y_1^h = Y_{n1}^{\text{TE}} \coth \Gamma_{n1} h$$

$$Y_2^e = Y_{n2}^{\text{TM}} \coth \Gamma_{n2} (a - h) \quad Y_2^h = Y_{n2}^{\text{TE}} \coth \Gamma_{n2} (a - h)$$

The final step is to translate the relationships (39) in the  $(u, v)$  coordinate system to the  $(y, z)$  coordinates using the transformation (37). This yields the desired relationships in the spectral domain in the form of (21), where

$$\tilde{Z}_{yy}(\alpha_n) = \tilde{Z}^e(\alpha_n) \sin^2 \theta + \tilde{Z}^h(\alpha_n) \cos^2 \theta \quad (41a)$$

$$\tilde{Z}_{yz}(\alpha_n) = \tilde{Z}_{zy}(\alpha_n) = \left[ \tilde{Z}^e(\alpha_n) - \tilde{Z}^h(\alpha_n) \right] \sin \theta \cos \theta \quad (41b)$$

$$\tilde{Z}_{zz}(\alpha_n) = \tilde{Z}^e(\alpha_n) \cos^2 \theta + \tilde{Z}^h(\alpha_n) \sin^2 \theta \quad (41c)$$

Thus we arrive at the desired spectral domain impedance functions. The approach is easily generalized to more complicated multi-layer geometries using the equivalent circuit concept, and greatly simplifies the construction of the impedance functions. For example, note that if the top wall is removed so there is no shielding on the microstrip, this is easily accounted for by making the top transmission-line in the equivalent circuit semi-infinite, such that

$$Y_2^e = Y_{n2}^{\text{TM}} \quad Y_2^h = Y_{n2}^{\text{TE}} \quad (42)$$

For some structures such as fin-line [3] or coplanar waveguide, it is more convenient to use an admittance matrix which expresses the interfacial currents in terms of the interfacial fields,

*i.e.* the inverse of (39). However the equivalent circuit approach is easily extended to that situation, where the interfacial fields are represented as voltage sources in the equivalent circuit.

As presented, the method is restricted to the computation of dispersion characteristics for the transmission line. In order to compute the characteristic impedance, the fields in each region must still be derived by a field-matching procedure. Some techniques for generalizing the field-mapping procedure to more complicated multi-layer systems, using a transfer matrix approach, are described in [4].

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