HOMEWORK #3

3 METHOD OF MOMENTS SOLUTION OF HALLEN'S EQUATION

Hallen's equation was derived in class as:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} I(z') \frac{e^{-jkR}}{R} dz' = -j\frac{2\pi}{\eta_0} V_g \sin k_0|z| - C\cos k_0 z \tag{1}$$

where

$$R = \sqrt{a^2 + (z - z')^2}.$$

Hallen's equation applies to the center-fed dipole. From the symmetry of the problem, we expect the current on each half of the dipole to be the same

$$I(z) = I(-z)$$

and so the integral can be rewritten as

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} I(z') \frac{e^{-jkR}}{R} dz' = \int_{0}^{\frac{L}{2}} I(z') \underbrace{\left[\frac{e^{-jk_0 R_1}}{R_1} + \frac{e^{-jk_0 R_2}}{R_2} \right]}_{G(z,z')} dz' \tag{2}$$

where

$$\begin{aligned} R_1 &= \sqrt{a^2 + (z - z')^2} \\ R_2 &= \sqrt{a^2 + (z - z')^2} \end{aligned}$$

Following the Method of Moments procedure, we expand the current in a set of basis functions

$$I(z) = \sum_{n=1}^{N} I_n \Phi_n(z) \tag{3}$$

so that

$$\sum_{n=1}^{N} I_n \int_0^{\frac{L}{2}} \Phi_n(z') G(z,z')\, dz' + C\cos k_0 z = \frac{-jV_g}{60} \sin k_0|z| \tag{4}$$

Note that there are $N+1$ unknowns in this equations. Multiply both sides by a set of $N+1$ weighting functions $w_m(z)$ and integrate to give

$$\sum_{n=1}^{N} I_n Z_{mn} + C Y_m = V_m \tag{5}$$

where

$$
\begin{aligned}
Z_{mn} &= \int_0^{\frac{L}{2}} \int_0^{\frac{L}{2}} w_m(z) G(z,z') \Phi_n(z')\, dz'\, dz \\
Y_m &= \int_0^{\frac{L}{2}} w_m(z) \cos k_0(z)\, dz \\
V_m &= \frac{-jV_g}{60} \int_0^{\frac{L}{2}} w_m(z) \sin k_0|z|\, dz
\end{aligned}
$$

For notational convenience let $C = I_{N+1}$, so that

$$\sum_{n=1}^{N+1} I_n A_{mn} = V_m \tag{6}$$

where

$$A_{mn} = \begin{cases} Z_{mn} & \text{for } n \le N \\ Y_m & \text{for } n = N+1 \end{cases}$$

The solution of this matrix equation is

$$\overline{I} = \overline{\overline{A}}^{-1} \overline{V} \tag{7}$$

Far-Fields

In our discussion of wire antennas we found that a filamentary current along the z-axis gives rise to an electric field with a $\hat{\theta}$ component given by

$$E_\theta \propto \sin\theta \int_{-L/2}^{L/2} I(z') e^{jk_0 z' \cos\theta} dz' \tag{8}$$

Substituting (3) and exploiting the symmetry of the current distribution gives

$$E_\theta \propto 2 \sin\theta \sum_{n=1}^{N} I_n \int_0^{L/2} \Phi(z') \cos\left(k_0 z' \cos\theta\right)\, dz' \tag{9}$$

For simple basis functions, the integrals can usually be carried out analytically.

Pulse Basis and Point Matching

For point matching we enforce the equation at $N+1$ discrete points on the wire given by

$$z_n = (n-1)\Delta \qquad \Delta = \frac{L}{2N} \qquad n = 1 \dots N+1 \tag{10}$$

The weighting functions are then Dirac delta functions,

$$w_m(z) = \delta(z - z_m) \tag{11}$$

Pulse basis functions for this problem are defined by

$$\Phi_n(z) = \begin{cases} 1 & |z - z_n| \leq \frac{\Delta}{2} \\ 0 & \text{elsewhere} \end{cases} \qquad 0 < z < L/2 \tag{12}$$

The matrix parameters are then described by

$$Z_{mn} = \begin{cases} \displaystyle\int_0^{\frac{\Delta}{2}} G(z_m, z')\, dz' & \text{for } n = 1 \\ \displaystyle\int_{z_n - \frac{\Delta}{2}}^{z_n + \frac{\Delta}{2}} G(z_m, z')\, dz' & 1 < n \leq N \end{cases} \tag{13a}$$

$$V_m = \frac{-jV_g}{60} \sin k_0 z_m \tag{13b}$$

The input impedance is then $Z_{\text{in}} = V_g/I_0$, or simply $1/I_0$ if $V_g = 1$.

Using (9), the far-fields are given by

$$E_\theta \propto \sin\theta \sum_{n=1}^{N} I_n E_n(\theta) \tag{14}$$

where

$$E_n(\theta) = \begin{cases} \displaystyle\int_0^{\Delta/2} \cos\left(k_0 z' \cos\theta\right)\, dz' & n = 1 \\ \displaystyle\int_{z_n - \Delta/2}^{z_n + \Delta/2} \cos\left(k_0 z' \cos\theta\right)\, dz' & 1 < n \leq N \end{cases}$$

An attached *Mathematica* code computes the current distribution and radiation pattern for a wire antenna using this MoM technique.