

Stability of Mode Locked States of Coupled Oscillator Arrays

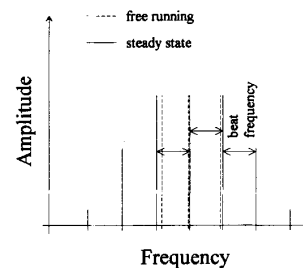
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Abstract—A method for analyzing beat frequency entrained (or mode locked) systems of coupled nonlinear oscillators is presented. Stable mode locked states are almost periodic oscillations that occur outside of the fundamental entrainment region of the array, and generate a periodic pulse train when the oscillator outputs are summed. The analysis relates the stability of the states to the relationship between the frequency pullings and the time average phases of the oscillators. The method is applied to three mode locked Van der Pol oscillators with arbitrary coupling time delay, and shows that the mode locking bandwidth is maximized for specific values of coupling delay and oscillator nonlinearity parameter.

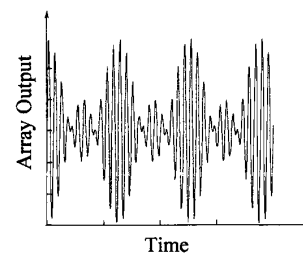
I. INTRODUCTION

MILLIMETER-WAVE systems are increasingly in demand for compact high resolution radar systems and satellite communications. However, it is difficult to generate the necessary transmitter power at these frequencies using solid state devices, due to the small size of the devices. Coupled-oscillator systems provide a solution to the problem by coherently combining the radiated power of many small microwave or millimeter-wave oscillators. Thus high transmit power can be achieved while maintaining low device temperatures and high oscillator efficiencies [1].

In practice, any useful operational mode of a system of coupled oscillators where coherent power combining is desired must exhibit some type of stable periodicity. This requires a frequency entrainment condition to exist between all of the oscillators in order to maintain the periodic state when the system is perturbed. For example, arrays of oscillators entrained to a common frequency have been used for electronically scanned transmitting arrays [2]. In this case the analysis is relatively straightforward since the steady-state solution is formally periodic and the dynamic equations autonomous. However, for other types of entrainment the solutions are not periodic but are almost periodic, in which the carrier is periodically modulated at a noncommensurate frequency [3]. One of the authors has experimentally verified [4] that a system of coupled oscillators will settle into a stable “mode locked” state with evenly spaced steady-state frequencies when the uncoupled, or free running, frequencies are nearly evenly spaced. The frequency spectrum of this state is depicted in Fig. 1(a), and the time domain waveform in Fig. 1(b) for three oscillators. Frequency entrainment occurs between an oscillator fundamental and



(a)



(b)

Fig. 1. Mode locking in a three element array. (a) Frequency spectrum showing free-running frequency distribution and stable steady-state mode locking spectrum. (b) Combined output waveform from array in the steady state.

the mixing products created by neighboring oscillators and maintains the even spectral spacing. This situation is referred to as “mode locking” after the analogous technique used to generate picosecond duration light pulses in laser systems [5]. Mode locking is similar to the better known phenomenon of entrainment to a common frequency. For example, if any element in the mode locked array is slightly detuned, the equal frequency spacing is maintained; however, the relative phase shifts between elements varies. Such a system may be useful at microwave and millimeter-wave frequencies for pulsed transmitters or automatic beam scanning [6].

The following analysis will be separated into two parts. The first is the derivation of a system of algebraic equations that determine the steady-state frequencies and average phases of the periodic states. This step will not be developed in detail here since many methods can be found in texts on perturbation theory and nonlinear oscillations, although an example will be given in Section III [7]. The second step is to assess the stability of the periodic states and is the focus of this paper. Finally, an array of Van der Pol oscillators will be analyzed using the presented technique.

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II. ANALYSIS OF PERIODIC STATES

The main restrictions on the following analysis are that the differential equation describing the dynamics of a single oscillator, neglecting coupling, is second order and nearly linear, and that the coupling between oscillators is weak. With these assumptions, the system of differential equations that (approximately) describes the time evolution of the amplitudes and phases of N coupled oscillators can be derived using the method of averaging [8] and has the general form

$$\begin{aligned} \frac{d\phi}{dt} &= \beta + \varepsilon f(\phi, A, t) \\ \frac{dA}{dt} &= g(\phi, A, t) \end{aligned} \quad (1)$$

where ϕ , A , and β are N -element vectors, f and g are vector functions, t is the time parameter, and ε is a small dimensionless parameter that represents the coupling strength between oscillators. The ϕ correspond to time dependent phases, the A to time dependent amplitudes, and the β to the frequency pulling variables, or the difference between the uncoupled frequency ($\varepsilon = 0$) and the steady-state frequency for each oscillator. An almost periodic state of the original system (i.e., before the averaging method is applied) corresponds to a periodic state of (1). The frequency pulling for each oscillator is time independent and is a function of physical constants such as capacitance or inductance as well as the steady-state frequency. It is important to note that in (1) the steady-state phases ϕ are bounded and periodic since the steady-state frequencies are included implicitly in the frequency pulling terms β .

A. Determination of the Frequencies and Phases

When the free-running frequencies are nearly evenly spaced, the system can settle into a mode locked state where the oscillator frequencies are *exactly* evenly spaced by an amount that will be called the *beat frequency* [see Fig. 1(a)]. As mentioned previously, the frequencies and phases of a mode locked state of (1) can be found using a perturbation analysis such as the Poincaré-Lindstedt method [7], [8]. The result is a set of algebraic equations relating the frequency pulling variables to the time average phases of the oscillators:

$$\beta = F(\phi^{avg}). \quad (2)$$

Assuming a solution exists for a given set of free running frequencies, solving the N equations (2) provides the beat frequency, one of the steady-state frequencies, and $N - 2$ average phases. Since both the envelope and the carrier of the combined array output are periodic, two of the average phases are arbitrary. If (2) has a solution for a given set of oscillator tunings, then a mode locked state exists for that set of tunings. In addition to providing information about the existence of periodic states, we will show that equations (2) also provide information about the stability of the states.

B. Stability of Periodic States

The stability of a periodic solution can be tested by applying small perturbations to the phase variables and observing the response of the resulting linear system. However, this approach

leads to a linear system with periodic coefficients, and the stability of such systems depends on the Floquet multipliers which are, in general, difficult to determine. Under certain conditions, which are often satisfied in practical systems, we can find the constant coefficients of an approximate "averaged" system for the phase perturbations. This will allow us to investigate the stability of periodic solutions using the well known techniques of linear algebra applied to systems with constant coefficients.

The phase variables can either be perturbed directly, leaving the β unchanged, or by perturbing the free running frequencies. We will show how the relation between these two types of perturbations can be used to determine the coefficients of the averaged system. Treating the first type of perturbation, expand the ϕ 's and the A 's about a periodic solution:

$$\begin{aligned} \frac{d}{dt}(\phi^p + \delta) &= \beta^p + \varepsilon f(\phi^p + \delta, A^p + \alpha, t) \\ \frac{d}{dt}(A^p + \alpha) &= g(\phi^p + \delta, A^p + \alpha, t) \end{aligned} \quad (3)$$

where δ and α are the perturbations and the superscript "p" denotes the periodic solution. The dynamic equations of the perturbation, or the variational equations, result from a first order expansion about the periodic solution:

$$\begin{aligned} \frac{d\delta_n}{dt} &= \varepsilon \sum_m \left[\left. \frac{\partial f_n}{\partial \phi_m} \right|_{\phi_m^p} \delta_m + \left. \frac{\partial f_n}{\partial A_m} \right|_{A_m^p} \alpha_m \right] \\ \frac{d\alpha_n}{dt} &= \sum_m \left[\left. \frac{\partial g_n}{\partial \phi_m} \right|_{\phi_m^p} \delta_m + \left. \frac{\partial g_n}{\partial A_m} \right|_{A_m^p} \alpha_m \right]. \end{aligned} \quad (4)$$

Note that the coefficients multiplying the perturbations are periodic functions of time.

When ascertaining the stability of most systems of coupled oscillators, it is sufficient to study the system response to initial values of the phase perturbation variables only, maintaining the initial amplitude perturbations at zero [8]. This is sufficient as long as the transient responses of the amplitude variables decay quickly. The second equation in (4) is a nonhomogeneous system of linear differential equations for α where the forcing functions are superpositions of the variables δ . The solution, assuming it exists, can be written as the sum of a transient homogeneous solution and the particular solution. In addition, the particular solution can always be written as a linear combination of the phase variables. Thus the general solution has the form

$$\alpha_n(t) = h_n^{\text{homo}}(t) + \sum_m q_{nm}(t) \delta_m(t). \quad (5)$$

We shall assume the transient part dies out quickly and therefore does not appreciably affect the phase dynamics. Neglecting the homogeneous portion, inserting (5) into the first of (4) eliminates the amplitude perturbation variables. The dynamic equations for the phase perturbations become

$$\begin{aligned} \frac{d\delta_n}{dt} &= \varepsilon \sum_m \left[\left. \frac{\partial f_n}{\partial \phi_m} \right|_{\phi_m^p} \delta_m + \left. \frac{\partial f_n}{\partial A_m} \right|_{A_m^p} \sum_l q_{ml} \delta_l \right] \\ &= \varepsilon \sum_m \left[\left. \frac{\partial f_n}{\partial \phi_m} \right|_{\phi_m^p} + \sum_l \left. \frac{\partial f_n}{\partial A_l} \right|_{A_l^p} q_{lm} \right] \delta_m. \end{aligned} \quad (6)$$

The partial derivatives are still evaluated at the periodic solution but this is not explicitly shown for notational convenience. The derivatives of the phase perturbations are proportional to the small parameter ε so the phases must vary slowly and we can “average” the equations over time in such a way that any fast variations are averaged out but the slow variations are retained. Formally, we are applying the averaging method of Krylov and Bogoliubov [3], [8]. Using brackets to represent an averaging (or filtering) operation, the slowly varying “average” value of the phase n th phase perturbation is denoted as

$$d_n(t) = \langle \delta_n(t) \rangle \quad (7)$$

and the equivalent averaged system with constant coefficients is:

$$\begin{aligned} \frac{d}{dt} d_n &= \varepsilon \sum_m \left[\left\langle \frac{\partial f_n}{\partial \phi_m} \right\rangle + \sum_l \left\langle \frac{\partial f_n}{\partial A_l} q_{lm} \right\rangle \right] d_m \\ &= \varepsilon \sum_m C_{nm} d_m. \end{aligned} \quad (8)$$

For a complete justification of this method see [3], [8], and [9]. Fortunately, the coefficients C_{nm} need not be evaluated directly but can be calculated from the frequency pulling variables β , as we will now show.

Consider an infinitesimal change $d\beta$ to the frequency pulling of a periodic state and allow the system to settle to a new periodic state. The difference between the new and old phases is determined by

$$\frac{d}{dt} (d\phi_n^p) = d\beta_n + \varepsilon \sum_m \left[\frac{\partial f_n}{\partial \phi_m} d\phi_m^p + \frac{\partial f_n}{\partial A_m} dA_m^p \right]. \quad (9)$$

These partials are evaluated at the original periodic values, that is, before β was perturbed. Since the new solution to (9) is also periodic, its time average over one period T must be constant and, therefore, the time average of the derivative of the phase perturbation must vanish:

$$\frac{1}{T} \int_0^T \left[\frac{d}{dt} (d\phi^p) \right] dt = 0 \quad (10)$$

so,

$$d\beta_n = -\varepsilon \sum_m \left\langle \frac{\partial f_n}{\partial \phi_m} d\phi_m^p + \frac{\partial f_n}{\partial A_m} dA_m^p \right\rangle. \quad (11)$$

Using the expression in (5) for the amplitudes we have

$$d\beta_n = -\varepsilon \sum_m \left\langle \left[\frac{\partial f_n}{\partial \phi_m} + \sum_l \frac{\partial f_n}{\partial A_l} q_{lm} \right] d\phi_m^p \right\rangle. \quad (12)$$

The phase perturbation $d\phi_m^p$ is a time dependent quantity that satisfies (9). Since its derivative is proportional to the small quantity ε it must vary slowly and we can therefore approximate the average value of the product of the bracketed terms with $d\phi_m^p$ in (12) as the product of the average values [note that the differential change in β in (9) can be neglected when determining the smallness of $d\phi_m^p$ because the average

value of the summation must cancel $d\beta$ in order to maintain a bounded $d\phi_m^p$]. The approximate expression for $d\beta$ becomes

$$d\beta_n = -\varepsilon \sum_m \left[\left\langle \frac{\partial f_n}{\partial \phi_m} \right\rangle + \sum_l \left\langle \frac{\partial f_n}{\partial A_l} q_{lm} \right\rangle \right] d\phi_m^{avg} \quad (13)$$

where

$$d\phi_m^{avg} = \frac{1}{T} \int_0^T [d\phi_m^p] dt \quad (14)$$

and T is one period.

Thus the coefficients in (8) can be identified as

$$C_{nm} = -\frac{\partial \beta_n}{\partial \phi_m^{avg}}. \quad (15)$$

This simple relation allows us to evaluate the coefficients of the averaged system by simply differentiating the expressions for the frequency pulling of each oscillator. The stability of a periodic state can then be determined by computing the eigenvalues of the matrix of coefficients. It is interesting to note that this result is exact for the case of an autonomous set of equations linearized about a fixed point. It is satisfying to see that the result represents the “average” values for the nonautonomous system under the stated conditions.

III. EXAMPLE: THREE MODE LOCKED VAN DER POL OSCILLATORS

To illustrate the proposed method, consider three Van der Pol oscillators that are identical in every respect except for their free running frequencies. Also assume each is coupled to its nearest neighbors through the damping terms and include the possibility of a coupling time delay. If the free running frequencies are almost evenly spaced, the system will lock to a mode in which the steady-state frequencies are exactly evenly spaced. Any small change to the free running frequencies will produce a corresponding shift in the average phases of the oscillators. This phenomenon is analogous to the injection locking of two oscillators to a common frequency [10] and to mode locking in pulsed laser physics [5]. The dynamic equations for the array are

$$\begin{aligned} \dot{x}_1 + \omega_{o1}^2 x_1 - 2\mu\omega_a(1 - 4x_1^2)\dot{x}_1 &= 2\lambda\omega_a \dot{x}_2(t - \tau_d) \\ \dot{x}_2 + \omega_{o2}^2 x_2 - 2\mu\omega_a(1 - 4x_2^2)\dot{x}_2 &= 2\lambda\omega_a [\dot{x}_1(t - \tau_d) \\ &\quad + \dot{x}_3(t - \tau_d)] \\ \dot{x}_3 + \omega_{o3}^2 x_3 - 2\mu\omega_a(1 - 4x_3^2)\dot{x}_3 &= 2\lambda\omega_a \dot{x}_2(t - \tau_d) \end{aligned} \quad (16)$$

where ω_{on} are the free running frequencies, τ_d is a coupling time delay, μ is the nonlinearity parameter, λ is the coupling parameter, and ω_a is the half bandwidth of the oscillator. The form of coupling indicated in (16) occurs in radiatively coupled microwave oscillator arrays [10]. Assuming that μ and λ are both small, the variable x_n can be written in terms of its slowly varying amplitude and phase as

$$x_n(t) = A_n(t) \cos[\omega_n t + \phi_n(t)]. \quad (17)$$

Again, the frequency ω_n is the steady-state frequency so that in the mode locked state $\phi_n(t)$ is periodic. Applying the transformation of Krylov-Bogoliubov and the method of

averaging gives the dynamic equations for the amplitudes and phases:

$$\begin{aligned} \frac{dA_n}{d\tau} &= \eta(1 - A_n^2)A_n + \varepsilon[A_{n-1} \cos(\tau + \phi_n - \phi_{n-1} + \Phi) \\ &\quad + A_{n+1} \cos(\tau + \phi_{n+1} - \phi_n - \Phi)] \\ \frac{d\phi_n}{d\tau} &= \beta_n - \varepsilon \left[\frac{A_{n-1}}{A_n} \sin(\tau + \phi_n - \phi_{n-1} + \Phi) \right. \\ &\quad \left. - \frac{A_{n+1}}{A_n} \sin(\tau + \phi_{n+1} - \phi_n - \Phi) \right] \end{aligned} \quad (18)$$

where $n = 1, 2, \dots, N$ and the normalized parameters are defined as

$$\begin{aligned} \eta &= \mu \frac{\omega_a}{\omega_b}, \\ \varepsilon &= \lambda \frac{\omega_a}{\omega_b}, \\ \beta_n &= \frac{\omega_{on} - \omega_n}{\omega_b}, \\ \tau &= \omega_b t, \\ \Phi &= \omega_o \tau_d \approx \text{const.} \end{aligned} \quad (19)$$

The beat frequency ω_b is the steady-state frequency difference between adjacent oscillators and is identical for all adjacent oscillator pairs, and the frequency ω_o is the array "center" frequency which can be taken as the average of the element frequencies for small bandwidth systems. Equations (18) apply to all elements provided any quantities with subscripts zero or $N + 1$ are set to zero.

Applying the Poincaré-Lindstedt perturbation method gives the frequency pulling variables as functions of the average phases. We begin by expanding all unknown variables in power series of the small coupling parameter ε ,

$$\begin{aligned} A_n &= A_n^{(0)} + \varepsilon A_n^{(1)} + \varepsilon^2 A_n^{(2)} + \dots \\ \phi_n &= \phi_n^{(0)} + \varepsilon \phi_n^{(1)} + \varepsilon^2 \phi_n^{(2)} + \dots \\ \beta_n &= \beta_n^{(0)} + \varepsilon \beta_n^{(1)} + \varepsilon^2 \beta_n^{(2)} + \dots \end{aligned} \quad (20)$$

Inserting these into the dynamic (18) and equating like powers of ε gives an infinite set of differential equations that can be solved recursively. The zeroth order equations are

$$\begin{aligned} \frac{dA_n^{(0)}}{d\tau} &= \eta(1 - A_n^{(0)2})A_n^{(0)}, \\ \frac{d\phi_n^{(0)}}{d\tau} &= \beta_n^{(0)} \end{aligned} \quad (21)$$

and the steady-state solutions can be found by enforcing the periodicity of amplitudes and phases. Thus,

$$\begin{aligned} A_n^{(0)} &= 1, \\ \phi_n^{(0)} &= \text{constant}, \\ \beta_n^{(0)} &= 0. \end{aligned} \quad (22)$$

Using these results in the first order system gives

$$\begin{aligned} \frac{dA_n^{(1)}}{d\tau} &= -2\eta A_n^{(1)} + [\cos(\tau + \Delta\phi_{n-1} + \Phi) \\ &\quad + \cos(\tau + \Delta\phi_n - \Phi)] \end{aligned}$$

$$\begin{aligned} \frac{d\phi_n^{(1)}}{d\tau} &= \beta_n^{(1)} - [\sin(\tau + \Delta\phi_{n-1} + \Phi) \\ &\quad - \sin(\tau + \Delta\phi_n - \Phi)] \end{aligned} \quad (23)$$

where $\Delta\phi_n = \phi_{n+1}^{(0)} - \phi_n^{(0)}$. As before, enforcing periodicity gives the steady-state values:

$$\begin{aligned} A_n^{(1)} &= \frac{1}{\Gamma^2} [\sin(\tau + \Delta\phi_{n-1} - \Phi) + 2\eta \cos(\tau + \Delta\phi_{n-1} + \Phi) \\ &\quad + \sin(\tau + \Delta\phi_n - \Phi) + 2\eta \cos(\tau + \Delta\phi_n - \Phi)] \\ \phi_n^{(1)} &= \cos(\tau + \Delta\phi_{n-1} + \Phi) - \cos(\tau + \Delta\phi_n - \Phi) \\ \beta_n^{(1)} &= 0 \end{aligned} \quad (24)$$

where $\Gamma = 1/\sqrt{1 + (2\eta)^2}$. One can see from the above equations that the amplitude and phase perturbations are, to the first order, sinusoidal. The amplitudes and phases of the perturbations depend on the nonlinearity parameter, the coupling phase, and the zeroth order phases, which are approximately equal to the time average phases correct to the first order. Inserting the zeroth and first order solutions into the second order equations and enforcing periodicity we can find the second order steady-state frequency pulling variables. The result is

$$\begin{aligned} \beta_n &= \varepsilon^2 \left[\frac{\eta}{\Gamma^2} \sin(\Delta\Delta\phi_{n+2}) + \frac{1}{2} \left(1 - \frac{1}{\Gamma^2} \right) \cos(\Delta\Delta\phi_{n+2}) \right. \\ &\quad + \frac{2\eta}{\Gamma^2} \sin(2\Phi) + \frac{2\eta}{\Gamma^2} \sin(\Delta\Delta\phi_{n+1} - 2\Phi) \\ &\quad \left. + \frac{\eta}{\Gamma^2} \sin(\Delta\Delta\phi_n) - \frac{1}{2} \left(1 - \frac{1}{\Gamma^2} \right) \cos(\Delta\Delta\phi_n) \right] \end{aligned} \quad (25)$$

where $n = 1, 2, \dots, N$. As before, any variable with a subscript not in this range should be eliminated. For the three element array the pulling variables are

$$\begin{aligned} \beta_1 &= -\varepsilon^2 \left[\cos^2(\Phi) - \frac{1}{\Gamma^2} \sin^2(\Phi) - \frac{\eta}{\Gamma^2} \sin(2\Phi) \right. \\ &\quad \left. - \frac{\eta}{\Gamma^2} \sin(\Delta\Delta\phi_1) - \frac{1}{2} \left(1 - \frac{1}{\Gamma^2} \right) \cos(\Delta\Delta\phi_1) \right] \\ \beta_2 &= \varepsilon^2 \frac{2\eta}{\Gamma^2} [\sin(2\Phi) + \sin(\Delta\Delta\phi_1 - 2\Phi)] \\ \beta_3 &= \varepsilon^2 \left[\cos^2(\Phi) - \frac{1}{\Gamma^2} \sin^2(\Phi) + \frac{\eta}{\Gamma^2} \sin(2\Phi) + \frac{\eta}{\Gamma^2} \right. \\ &\quad \left. \cdot \sin(\Delta\Delta\phi_1) - \frac{1}{2} \left(1 - \frac{1}{\Gamma^2} \right) \cos(\Delta\Delta\phi_1) \right] \end{aligned} \quad (26)$$

and the second difference of the n th phase is defined as

$$\begin{aligned} \Delta\Delta\phi_1 &= \phi_3^{(0)} - 2\phi_2^{(0)} + \phi_1^{(0)} \\ &\cong \phi_3^{avg} - 2\phi_2^{avg} + \phi_1^{avg}. \end{aligned} \quad (27)$$

Equations (26) are a result of the nonlinear interaction of the amplitude and phase perturbations with the signals from neighboring oscillators. Whether or not the perturbations contribute to mode locking depends on the relative phases of the perturbations. A close look at the source of the various terms in (26) shows that the sine terms result only from the amplitude perturbations, and we will see below that only these terms contribute to mode locking. The cosine terms shift the end element frequencies in equal but opposite directions and

thus maintain equal spacing of the spectral components. One can show that the cosine phases of the amplitude perturbations in (24) produce the sine terms in (26) and these terms are maximized for $\eta = 1/2$. In addition, a coupling phase of 90° produces an optimum relative phase of the amplitude perturbations. These optimum values will become clearly evident when we derive the size of the mode locking region below.

The goal is to determine the range of free running frequencies for which stable mode locked states exist. If our concern is the phase distribution and stability for a given set of ω_{on} , and not the beat frequency and spectral location, we only need to consider the second difference of the β_n :

$$\begin{aligned}\Delta\Delta\beta &= \frac{\omega_{o3} - 2\omega_{o2} + \omega_{o1}}{\omega_b} \\ &= \frac{2}{\omega_b} \left(\frac{\omega_{o3} + \omega_{o1}}{2} - \omega_{o2} \right) \\ &= \varepsilon^2 \frac{2\eta}{\Gamma^2} [-\sin(2\Phi) + \sin(\Delta\Delta\phi_1) \\ &\quad - 2\sin(\Delta\Delta\phi_1 - 2\Phi)].\end{aligned}\quad (28)$$

One can see that $\Delta\Delta\beta$ is a measure of the proximity of the free running frequency of the center oscillator to the midpoint between the free running frequencies of the outer oscillators. Rewriting (28) as

$$\begin{aligned}\Delta\Delta\beta &= \varepsilon^2 \frac{2\eta}{\Gamma^2} \left\{ \sqrt{5 - 4\cos(2\Phi)} \right. \\ &\quad \cdot \sin \left[\Delta\Delta\phi_1 + \tan^{-1} \frac{2\sin(2\Phi)}{1 - 2\cos(2\Phi)} \right] \\ &\quad \left. - \sin(2\Phi) \right\}\end{aligned}\quad (29)$$

reveals the maximum and minimum values of $\Delta\Delta\beta$. The total range of $\Delta\Delta\beta$ is

$$\Delta\Delta\beta|_{\max} - \Delta\Delta\beta|_{\min} = 2\varepsilon^2 \frac{2\eta}{\Gamma^2} \sqrt{5 - 4\cos(2\Phi)}. \quad (30)$$

By fixing the free running frequencies of the two end oscillators and varying only the center, we can obtain an expression for the width of the band of possible values of ω_{o2} for which a solution of (29) exists. The result is

$$\omega_{o2}|_{\max} - \omega_{o2}|_{\min} = \omega_b \varepsilon^2 \frac{2\eta}{1 + (2\eta)^2} \sqrt{5 - 4\cos(2\Phi)}. \quad (31)$$

As mentioned before, this band is maximum when the coupling phase is $\Phi = 90^\circ$ and the nonlinearity parameter $\eta = 1/2$ because these parameter values produce the optimum phasing of the amplitude perturbations. There is no guarantee that a state is stable throughout the range indicated by (31), but the stability of the states can easily be determined using the results of Section II.

The coefficient for the averaged system can be easily found using (15) applied to (29). It should be noted that the original system of the form shown in (15) is of dimension three. However, two of the phase variables are arbitrary since the carrier and the envelope are both periodic functions of time so two of the eigenvalues of the matrix C are zero. If one

forms the stability matrix using the second differences of the frequency and phase variables the system dimension is reduced by two: the two zero eigenvalues are removed, and the third eigenvalue remains unchanged. For the three element array this procedure gives:

$$\begin{aligned}C_{11} &= -\frac{d(\Delta\Delta\beta)}{d(\Delta\Delta\phi_1)} \\ &= -\varepsilon^2 \frac{2\eta}{1 + (2\eta)^2} \left\{ \sqrt{5 - 4\cos(2\Phi)} \right. \\ &\quad \cdot \cos \left[\Delta\Delta\phi_1 + \tan^{-1} \frac{2\sin(2\Phi)}{1 - 2\cos(2\Phi)} \right] \left. \right\}.\end{aligned}\quad (32)$$

The system is stable for values of $\Delta\Delta\phi_1$ that make this coefficient negative. Thus the stability condition is

$$\left| \Delta\Delta\phi_1 + \tan^{-1} \frac{2\sin(2\Phi)}{1 - 2\cos(2\Phi)} \right| < \frac{\pi}{2}. \quad (33)$$

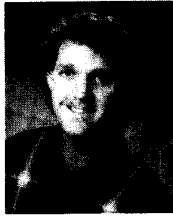
Every value of $\Delta\Delta\beta$ that exists corresponds to a stable state, however the stability condition above ensures that only a single stable state exists for a given set of free running frequencies since over the range of phases satisfied by (33) the sine function in (29) is single valued. The results here are similar to the results for two oscillators entrained at the same frequency.

IV. CONCLUSION

The proposed method for analyzing almost periodic oscillations by deriving the frequency pulling variables as functions of the average phases of the oscillators utilizes the averaging technique of Krylov and Bogoliubov. The significance of the present effort is in the physical interpretation of the analysis, especially the relationship between perturbations of the phases themselves and phase perturbations resulting from oscillator detuning. The Poincaré-Lindstedt perturbation method coupled with the analysis presented here gives a great deal of insight into the nature of mode locking phenomena.

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