# The Dynamics of Coupled Oscillator Phase Control

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Arrays of coupled oscillators have been proposed as means of realizing high power rf sources via coherent spatial power combining.[1][2] In such applications, a uniform phase distribution over the aperture is usually desired. However, it has been shown that by detuning some of the oscillators away from the oscillation frequency of the ensemble of oscillators, one may achieve other useful aperture phase distributions.[3] Of particular interest among those achievable are linear phase distributions because these result in steering of the output rf beam away from the broadside direction. The theory describing the behavior of such arrays of coupled oscillators is quite complicated since the phenomena involved are inherently nonlinear. However, a simplified theory has been developed which facilitates intuitive understanding.[4] This simplified theory is based on a "continuum model" in which the aperture phase is represented by a continuous function of the aperture coordinates. A challenging aspect of the development of this theory is the derivation of appropriate boundary conditions at the edges or ends of the array.

We begin by reviewing the nonlinear equations describing the behavior of an array of loosely coupled oscillators.[2] The behavior of the phase of a single oscillator injection locked to an input signal,

$$V_{inj} = A_{inj} e^{j(\omega_{inj} t + \psi_{inj})} = A_{inj} e^{j\theta_{inj}} \tag{1}$$

can be described by the following differential equation.

$$\frac{d\theta}{dt} = \omega_o + \Delta\omega_{lock} \sin(\theta_{inj} - \theta) \tag{2}$$

where $6 = \omega t + \phi$ , $\phi$ is the phase of the oscillator oscillating at frequency, $\omega$,

$$\Delta\omega_{lock} = \frac{\omega_o}{2Q}\frac{A_{inj}}{A} \tag{3}$$

the locking bandwidth which is inversely proportional to the Q of the oscillator and A, the amplitude of the oscillation. For a linear array of N coupled oscillators, the injection signals are just the outputs of the other oscillators and the phase of the $i^{th}$ oscillator is described by a differential equation of the form,

$$\frac{d\theta_i}{dt} = \omega_o - \frac{\omega_i}{2Q}\sum_{j=1}^{N} \varepsilon_{ij}\frac{A_j}{A_i}\sin(\Phi_{ij} + \theta_i - \theta_j) \tag{4}$$

and $\varepsilon_{ij}e^{j\Phi_{ij}}$ is the coupling between oscillators i and j. This system of equations, one equation for each value of i, has a coefficients matrix with zero determinant, a manifestation of the arbitrary nature of the phase reference. York has suggested that this may be remedied by dealing with phase differences between neighboring oscillators. Thus, we write,

$$\frac{d(\Delta\phi_i)}{dt} = (\omega_{tune,i+1} - \omega_{tune,i}) - \Delta\omega_{lock,ij}\sum_{\substack{j=i+1\\ j=i-1}}(\Delta\phi_i - \Delta\phi_j) \tag{5}$$

where,

$$\Delta\phi_i = \phi_{i+1} - \phi_i \rightarrow \frac{\partial\phi}{\partial x} = \tilde{\psi} \tag{6}$$

the phase gradient. Consider an infinitely long linear array in which the center oscillator tuning frequency is increased by an amount C from the ensemble frequency at t=0. Limiting the coupling to nearest neighbors and taking the continuum limit (i becomes a continuous variable, x), results in,

$$\frac{\partial^2\tilde{\psi}}{\partial x^2} - \frac{\partial\tilde{\psi}}{\partial\tau} = -\frac{1}{\Delta\omega_{lock}}\frac{\partial\omega_{tune}}{\partial x} = -Cu(\tau)\delta'(x) \tag{7}$$

where $\Delta\omega_{lock}$ is the mutual locking bandwidth of the coupled oscillators and includes the magnitude of the coupling constant, $\varepsilon_{ij}$, $\tau$ is time multiplied by the locking bandwidth, while $\omega_{tune}(x)$ gives the oscillator tuning frequencies. This equation may be solved via Laplace transformation with respect to time resulting in,

$$\phi(x,\tau)=C\left[2\sqrt{\frac{\tau}{\pi}}e^{-\frac{x^2}{4\tau}}-|x|erfc\left(\frac{|x|}{2\sqrt{\tau}}\right)\right]u(\tau) \tag{8}$$

Note that this diverges as the square root of time as time approaches infinity. However, the frequency function, obtained by time differentiation, converges to the original ensemble frequency as one over the square root of time which is consistent with the fact that detuning one of the infinite number of oscillators does not change the ensemble frequency; i.e., the average of the tuning frequencies.

Consider now a finite length array extending from -a to a along **x**. The boundary conditions are derived by writing the equations of (9) for the oscillators at the ends of the array and proceeding to the continuum limit. This leads to,

$$\left.\frac{\partial\tilde{\psi}}{\partial\tau}\right|_{-\left(a-\frac{1}{2}\right)}=\frac{1}{\Delta\omega_{lock}}\left.\frac{\partial\omega_{tune}}{\partial x}\right|_{-\left(a-\frac{1}{2}\right)}+\left.\frac{\partial\tilde{\psi}}{\partial x}\right|_{-(a-1)}-\tilde{\psi}\Big|_{-\left(a-\frac{1}{2}\right)} \tag{10}$$

$$\left.\frac{\partial\tilde{\psi}}{\partial\tau}\right|_{\left(a-\frac{1}{2}\right)}=\frac{1}{\Delta\omega_{lock}}\left.\frac{\partial\omega_{tune}}{\partial x}\right|_{\left(a-\frac{1}{2}\right)}-\left.\frac{\partial\tilde{\psi}}{\partial x}\right|_{(a-1)}-\tilde{\psi}\Big|_{\left(a-\frac{1}{2}\right)} \tag{11}$$

Solving (12) with these boundary conditions via Laplace transformation and integrating the resulting expression for the phase gradient gives the following for the transform of the aperture phase distribution.

$$f(x,s)=\frac{C}{2s^2}\left\{\frac{(s+1)\cosh\left[\sqrt{s}\left(a-\frac{1}{2}-|x|\right)\right]+\sqrt{s}\sinh\left[\sqrt{s}(a-1-|x|)\right]}{\cosh\left[\sqrt{s}(a-1)\right]+\frac{s+1}{\sqrt{s}}\sinh\left[\sqrt{s}\left(a-\frac{1}{2}\right)\right]}\right\} \tag{13}$$

This can be inverted via the calculus of residues in the form,

$$\phi(x,\tau)=\left[\frac{C\tau}{2a+1}+r_{-1}+\sum_{n=0}^{\infty}r_n e^{-\sigma_n\tau}\right]u(\tau) \tag{14}$$

where,

$$r_{-1}=\frac{C}{2(2a+1)}\left[\frac{(2a+1)}{2}-|x|\right]^2+K \tag{15}$$

indicating that the steady state frequency is shifted to the new average of the oscillator tuning frequencies and that the steady state phase distribution is quadratic. Figure 1 shows typical space-time behavior of the aperture phase of a finite array obtained from (16).

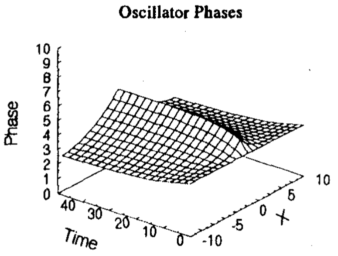

Figure 1. Typical space-time behavior of the phase of a finite length linear array of oscillators under step detuning of the center oscillator.

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